Instytut Matematyki

PhD in Mathematics

Algebraic hyperstructures in the model theory of valued fields

Candidate:
Alessandro Linzi

Supervisor:
Prof. Franz-Viktor Kuhlmann

Academic Year 2021-2022

To my grandparents:
Fiora, Luciano, Antonietta e Lino

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Contents

Introduction

Model theory is a part of mathematical logic born with the goal of investigating the foundations of mathematics. The relation between valued fields (defined below) and model theory dates back to the 1950s when A. Robinson (in [46]) proved the completeness of the elementary theory of algebraically closed valued fields (of fixed characteristic and residue characteristic). Some years later, several model theoretic arguments captured the attention of the mathematical community, especially of algebraists and number theorists. In their joint work [1] J. Ax and S. Kochen gave an important contribution to Artin's conjecture: a problem on the solvability of homogeneous diophantine equations over p-adic number fields. Ax and Kochen worked with valued fields and used tools coming from model theory. It is through this and other results that certain model theoretical methods and concepts begin to infiltrate algebra.

Let K be a field and Γ a linearly ordered abelian group (written additively). A surjective map

$$v : K \to \Gamma \cup \{\infty\}$$

is called a *(Krull) valuation on* K if it satisfies for all $x, y \in K$:

1. $v(x) = \infty$ if and only if $x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

Here, ∞ is a symbol such that $\gamma + \infty = \infty + \gamma = \infty$ and $\gamma < \infty$ for all $\gamma \in \Gamma$. If a valuation on a field K is given, then we will call (K, v) a *valued field*. We will further denote Γ by vK and call it the *value group of* (K, v). The value $v(x)$ of $x \in K$ will often be denoted by vx.

Let (K, v) be a valued field. The set

$$\mathcal{O}_v := \{x \in K \mid vx \geq 0\}$$

is a subring of K such that $x \in \mathcal{O}_v$ or $x^{-1} \in \mathcal{O}_v$ for all nonzero $x \in K$. A subring of a field satisfying this property is called a *valuation ring*. Any valuation ring is a local ring, i.e., it admits a unique maximal ideal. In the case of $\mathcal{O}_v$ this ideal is given by

$$\mathcal{M}_v := \{x \in K \mid vx > 0\}.$$

The *residue field* of (K, v), denoted by Kv, is the field $\mathcal{O}_v/\mathcal{M}_v$. The image $x + \mathcal{M}_v$ of $x \in \mathcal{O}_v$ under the canonical projection will be denoted by xv. The *residue characteristic* of a valued field (K, v) is the characteristic of the field Kv.

A valued field (K, v) is called *henselian* if for every monic polynomial $f \in \mathcal{O}_v[X]$ the following holds: if $b \in \mathcal{O}_v$ satisfies $vf(b) > 0$ and $vf'(b) = 0$, then f admits a root $a \in \mathcal{O}_v$ such that $av = bv$. Here, f' denotes the formal derivative of f. For more details on valued fields we refer the reader to [15].

The theorem traditionally attributed to Ax-Kochen (and Yu. Ershov who proved it independently in [16]) states that the (first order) properties of a henselian valued field of residue characteristic 0 are completely determined by the (first order) properties of its value group and residue field (see also [43]). This can be seen as a version of Robinson's result *relativized* to the value group and the residue field, where the class of valued fields under consideration is extended from algebraically closed valued fields to henselian valued fields of residue characteristic 0.

Quantifier elimination in valued fields is a very interesting subject on its own with many applications. We cite here [14] for a survey on some of these applications. For some history on the subject the reader may consult the introduction of [54]. J. Pas in [41] proved a well-known result on quantifier elimination in valued fields. It states that the theory of henselian valued fields of residue characteristic 0 admits quantifier elimination relative to the value group and the residue field in a language with an *angular component map* (for more details, we refer to [21]).

Another well-known result on quantifier elimination in valued fields is the theorem of A. MacIntyre [35], later generalized in [44], on quantifier elimination for p-adically closed fields (of fixed p-rank). In this case, the language of valued fields is extended with the so-called MacIntyre *power predicates*.

The story does not end here. In 1991 S. A. Basarab (see [3]) proposed his *mixed structures* relative to which he obtains quantifier elimination for henselian valued fields of characteristic 0. It is known that not all valued fields admit an angular component map. An example of a valued field which does not is given in [42]. In contrast, Basarab's structures are associated to any valued field of characteristic 0 and their description does not require anything more than the valuation. Nevertheless, as it can be expected, the mixed structures are way more involved to define and to handle than angular component maps or the power predicates of MacIntyre.

For this reason the work of Basarab was followed up by F.-V. Kuhlmann (see [26]) who succeded in simplifying the mixed structures by introducing the *structures of additive and multiplicative congruences* (in short: *amc-structures*). In doing so, he also made clear how his results (in the case of mixed characteristic) generalize the already known results for p-adically closed fields.

A further simplification of the amc-structures is given by Flenner (see [17]) with the so-called *leading term structures* (or *RV-structures*). Nowadays, the structures of Flenner are the structures most commonly used as tools for relative quantifier elimination in henselian valued fields of characteristic 0 for various purposes (see e.g. [50], [51], [20], [6], [7] and [19]).

Valued hyperfields are algebraic hyperstructures which were first studied in 1957 by M. Krasner in [24], in relation to valued fields. The prefix "hyper" stands for *multivalued.* A hyperstructure is an algebraic structure where one or more operations are multivalued. For example, in a hyperfield, the sum of two elements is a non-empty subset, rather than a single element. All fields are hyperfields where the sum of two elements always is a singleton. From this point of view, the concept of hyperfield is a generalization of the concept of field. Krasner noticed that the additive structure of a valued field (K, v) induces on the multiplicative quotient group $K^\times/1 + \mathcal{M}_v$ the structure of a hyperfield; a notion that he then axiomatized. Since the valuation v is well-defined on $K^\times/1 + \mathcal{M}_v$, Krasner was led to axiomatize valued hyperfields too.

Recently, J. Lee in [32] and Tolliver in [48] have obtained several interesting results using the valued hyperfields of Krasner. In particular, Lee showed a completeness result of Ax-Kochen-Ershov style relative to these valued hyperfields. In [13], B. Davvaz and A. Salasi gave a definition of valuation on hyperrings which generalizes the original one of Krasner. M. Marshall in [36] studied the definition of orderings and of positive cones in hyperfields. J. Jun in [28, 29] studied algebraic geometry over hyperrings and hyperfields. N. Bowler and T. Su in [4] have classified stringent hyperfields, which are hyperfields that are very close to fields. In [9, 10], A. Connes and C. Consani have studied hyperrings and hyperfields in relation to problems in number theory. In [53], O. Viro related hyperfields to tropical geometry. There is even a book on algebraic hyperstructures and their applications [11] and a book on general hyperring theory [12].

Our work has its origins in the observation that the valued hyperfields which Krasner associated to any valued field give an alternative way of seeing the leading term structures of Flenner. In the same time they provide more insight in other aspects of the structure theory of valued fields. Motivated by the work of Lee [32], we decided to systematically link these valued hyperfields with several other structures used for quantifier elimination in valued fields. This investigation considered the RV-structures, the amc-structures and angular component maps.

While working with angular component maps, we noticed some analogies between them and the initial forms in the *graded ring* associated to a valued field. This led us to link valued hyperfields with these structures too. Graded rings are structures commonly used by algebraic geometers. We show that they can be used also for the model theory of henselian valued fields of residue characteristic 0. Nevertheless, in the mixed characteristic case, they are not sufficient in order to obtain relative quantifier elimination (see Example 5.33). In this case, the RV-structures of Flenner, and thus the valued hyperfields, still have something to say (when higher levels are considered). This suggests that the valued hyperfields associated to a valued field might provide the algebraic geometers who work with graded rings with a new effective tool.

The text is organized as follows. In the first chapter, we give an overview of the model theoretic methods that are of interest for us. In the second chapter, the algebraic theory of valued hyperfields is introduced and developed. The third chapter is devoted to linking valued fields with these particular hyperstructures. The relations between the valued hyperfields associated to valued fields and the other structures mentioned above are studied in the fourth chapter. Finally, in the fifth and last chapter, we present the relative quantifier elimination results for henselian valued fields of characteristic 0.

Acknowledgements

The author would like to express his sincere gratitude to all the special people who contributed to this work, not only mathematically speaking. These include my supervisor Franz-Viktor Kuhlmann for his visions, infinite patience and world-opening advice, Katarzyna Kuhlmann and Hanna Stojałowska for all the interesting and helpful discussions we had while discovering hyperstructures, Piotr Błaszkiewicz for his thought-provoking questions and observations while we were working together, my colleague Hanna Ćmiel for her help with many technical problems an Italian may encounter while studying for his PhD in Poland, as well as Piotr Szewczyk for his numerous remarks during our discussions. These people were all active part of an enthusiastic research group which provided a wonderful setting for this work to be elaborated.

I would also like to thank my family who supported me from the start to the end with endless care and love.

Finally, thanks to the MSD program of the European Union and the University of Szczecin which financially supported me and my work in Poland.

Chapter 1

Model theory

In this chapter we give an introduction to all the concepts of model theory that will be used in this work. These include first order languages and structures, terms, formulae, elementary equivalence, theories, quantifier elimination and ultraproducts.

There are countless sources for all these concepts. Let us mention here a few such as [5], [43], [52] and [21]. We also used unpublished lecture notes on model theoretical algebra written by F.-V. Kuhlmann.

This chapter is intended to give precise definitions and fix notation. When proofs are not too involved or are not considered as standard, we have included them in order to provide a broader picture for the reader interested in details. When a proof has been omitted, reference for it is given.

The reader already familiar with model theory can skip this chapter at the beginning and use it as a reference when needed.

1.1 First order languages and structures

Definition 1.1. A *(first order) language* is a triple

$$\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$$

where

- $\mathcal{R}$ is a set and its elements are called *relation symbols*,
- $\mathcal{F}$ is a set and its elements are called *function symbols*,
- $\mathcal{C}$ is a set and its elements are called *constant symbols*.

Together with a language we will always consider implicitly given a countable set $\mathcal{V}$ of *variables* (usually denoted by $x, y, z, \ldots$ or $x_1, x_2, x_3, \ldots$), the *equality symbol* $\doteq$, the *logical connectives* $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$ as well as the *quantifiers* $\forall$ and $\exists$. Moreover, we will have auxiliary symbols such as "(" and ")" which are useful in writing formulae (see Section 1.2 below).

When we specify a language we will frequently use only one pair of set brackets, that is, we write $\mathcal{L} = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$. For example,

$$\mathcal{L}_{og} := \{<, +, -, 0\}$$

is the language of ordered groups.

All function and relation symbols come implicitly with an *arity* which is a non-negative integer specifying the number of inputs. In the example above, we have that $<$ is a relation symbol of arity 2, $+$ is a function symbol of arity 2 and $-$ is a function symbol of arity 1. We will sometimes call a relation (resp. function) symbol of arity $n \in \mathbb{N}$ an n-*ary* relation (resp. function) symbol.

Definition 1.2. Let $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$ be a language. A language $\mathcal{L}' = (\mathcal{R}', \mathcal{F}', \mathcal{C}')$ is called an *extension* of $\mathcal{L}$ if $\mathcal{R} \subseteq \mathcal{R}'$, $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{C} \subseteq \mathcal{C}'$.

For example the language of fields

$$\mathcal{L}_f = \{+, \cdot, -, ^{-1}, 0, 1\}$$

is an extension of the language of rings

$$\mathcal{L}_r = \{+, \cdot, -, 0\}.$$

Definition 1.3. For a given language $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, an $\mathcal{L}$-*structure* is a quadruple

$$\mathfrak{A} = (A, \mathcal{R}^{\mathfrak{A}}, \mathcal{F}^{\mathfrak{A}}, \mathcal{C}^{\mathfrak{A}})$$

where

- A is a non-empty set, called the *universe* of $\mathfrak{A}$,
- $\mathcal{R}^{\mathfrak{A}} = \{R^{\mathfrak{A}} \mid R \in \mathcal{R}\}$ is such that $R^{\mathfrak{A}} \subseteq A^n$ is a relation for every $R \in \mathcal{R}$ of arity $n \in \mathbb{N}$.
- $\mathcal{F}^{\mathfrak{A}} = \{f^{\mathfrak{A}} \mid f \in \mathcal{F}\}$ is such that $f^{\mathfrak{A}} : A^n \rightarrow A$ is a function for every $f \in \mathcal{F}$ of arity $n \in \mathbb{N}$.
- $\mathcal{C}^{\mathfrak{A}} = \{c^{\mathfrak{A}} \mid c \in \mathcal{C}\}$ is such that $c^{\mathfrak{A}}$ is an element of A for every $c \in \mathcal{C}$.

In what follows, the universe of the $\mathcal{L}$-structures $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ will always be denoted by A, B and C respectively.

If we have an $\mathcal{L}$-structure $\mathfrak{A}$ and we extend the language $\mathcal{L}$ to a language $\mathcal{L}'$, then it is always possible to interpret the new relation, function and constant symbols on A. For the new relation and function symbols we just choose any relations and functions on A of the same arity. Since we assume A to be nonempty, we can also choose arbitrary elements of A for the interpretation of the new constant symbols. A structure $\mathfrak{A}'$ thus obtained is called an *$\mathcal{L}'$-expansion* of $\mathfrak{A}$, and $\mathfrak{A}$ is called a *reduct* of $\mathfrak{A}'$.

Sometimes it is useful to give names to some or all elements of the universe of a structure, thereby extending the language. This is captured by the following definition.

Definition 1.4. Let $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$ be a language and $\mathfrak{A}$ be an $\mathcal{L}$-structure. For $S \subseteq A$, we denote by $\mathcal{L}(S)$ the constant extension of $\mathcal{L}$ given by

$$\mathcal{L}(S) := (\mathcal{R}, \mathcal{F}, \mathcal{C}_S)$$

where $\mathcal{C}_S := \mathcal{C} \cup \{c_a \mid a \in S\}$. Further, by $(\mathfrak{A}, S)$ we will denote the $\mathcal{L}(S)$-structure resulting as an expansion of $\mathfrak{A}$ by setting $c_a^{(\mathfrak{A},S)} = a$ for all $a \in S$.

Next we introduce the notion of a substructure. In what follows $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$ always denotes a language.

Definition 1.5. Take $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$. Then $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$ if $A \subseteq B$ and

- for all n-ary relation symbols $R \in \mathcal{R}$ we have that $R^{\mathfrak{B}} \cap A^n = R^{\mathfrak{A}}$,
- for all n-ary function symbols $f \in \mathcal{F}$ we have that $f^{\mathfrak{B}} \restriction A^n = f^{\mathfrak{A}}$,
- for all constant symbols $c \in \mathcal{C}$ we have that $c^{\mathfrak{B}} = c^{\mathfrak{A}}$.

Let us now define the notions of morphism, embedding and isomorphism.

Definition 1.6. Take $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$. A map $\sigma : A \to B$ is called a *morphism (from $\mathfrak{A}$ to $\mathfrak{B}$)* if it satisfies

(M1) $(a_1, \dots, a_n) \in R^{\mathfrak{A}} \implies (\sigma a_1, \dots, \sigma a_n) \in R^{\mathfrak{B}}$ for all $n \in \mathbb{N}$, all n-ary relation symbols $R \in \mathcal{R}$ and all $(a_1, \dots, a_n) \in A^n$.

(M2) $\sigma f^{\mathfrak{A}}(a_1, \dots, a_n) = f^{\mathfrak{B}}(\sigma a_1, \dots, \sigma a_n)$ for all $n \in \mathbb{N}$, all n-ary function symbols $f \in \mathcal{F}$ and all $(a_1, \dots, a_n) \in A^n$.

(M3) $\sigma c^{\mathfrak{A}} = c^{\mathfrak{B}}$ for all constant symbols $c \in \mathcal{C}$.

A morphism is called *strict* if it satisfies

(M1′) $(a_1, \dots, a_n) \in R^{\mathfrak{A}} \iff (\sigma a_1, \dots, \sigma a_n) \in R^{\mathfrak{B}}$ for all $n \in \mathbb{N}$, all n-ary relation symbols $R \in \mathcal{R}$ and all $(a_1, \dots, a_n) \in A^n$.

Further, we define an *embedding* to be an injective strict morphism and an *isomorphism* to be a surjective embedding. We say that two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* and write $\mathfrak{A} \simeq \mathfrak{B}$ if there exists an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Remark 1.7. Note that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ if and only if the inclusion $\iota : A \to B$ is an embedding from $\mathfrak{A}$ to $\mathfrak{B}$.

1.2 Terms and formulae

We now wish to introduce the notion of formula in a language $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$. First we have to define what is a term.

Definition 1.8. The set $T_{\mathcal{L}}$ of *$\mathcal{L}$-terms* is the smallest set such that

- $c \in T_{\mathcal{L}}$ for all $c \in \mathcal{C}$,
- $x \in T_{\mathcal{L}}$ for all $x \in \mathcal{V}$,
- if $t_1, \dots, t_n \in T_{\mathcal{L}}$ and $f \in \mathcal{F}$ is a function symbol of arity $n \in \mathbb{N}$, then $f(t_1, \dots, t_n) \in T_{\mathcal{L}}$.

Similarly, we define the set $T^c_{\mathcal{L}}$ of *constant $\mathcal{L}$-terms* to be the smallest set such that

- $c \in T^c_{\mathcal{L}}$ for all $c \in \mathcal{C}$,
- if $t_1, \dots, t_n \in T^c_{\mathcal{L}}$ and $f \in \mathcal{F}$ is a function symbol of arity $n \in \mathbb{N}$, then $f(t_1, \dots, t_n) \in T^c_{\mathcal{L}}$.

Lemma 1.9 (Induction on the complexity of terms)**.** *Let P be a property of $\mathcal{L}$-terms. Assume that $P(c)$ holds for all $c \in \mathcal{C}$ and that $P(x)$ holds for all $x \in \mathcal{V}$. Suppose in addition that $P(t_1), \dots, P(t_n)$ implies $P(f(t_1, \dots, t_n))$ for all $f \in \mathcal{F}$ of any arity $n \in \mathbb{N}$ and for all $t_1, \dots, t_n \in T_{\mathcal{L}}$. Then $P(t)$ holds for all $t \in T_{\mathcal{L}}$.*

Proof. Let $S \subseteq T_{\mathcal{L}}$ be the set of all constant $\mathcal{L}$-terms such that $P(t)$ holds. Then by assumption S satisfies

- $c \in S$ for all $c \in \mathcal{C}$,
- $x \in S$ for all $x \in \mathcal{V}$

- if $t_1, \ldots, t_n \in S$ and $f \in \mathcal{F}$ is a function symbol of arity $n \in \mathbb{N}$, then $f(t_1, \ldots, t_n) \in S$.

Thus, since $T_\mathcal{L}$ is the smallest set satisfying these properties, we obtain $T_\mathcal{L} \subseteq S$. Therefore, $S = T_\mathcal{L}$. □

Clearly, there is an analogous principle of induction on the complexity of constant terms.

Lemma 1.10 (Induction on the complexity of constant terms)**.** *Let P be a property of $\mathcal{L}$-terms. Assume that $P(c)$ holds for all $c \in \mathcal{C}$. Suppose in addition that $P(t_1), \ldots, P(t_n)$ implies $P(f(t_1, \ldots, t_n))$ for all $f \in \mathcal{F}$ of any arity $n \in \mathbb{N}$ and for all $t_1, \ldots, t_n \in T^c_\mathcal{L}$. Then $P(t)$ holds for all $t \in T^c_\mathcal{L}$.*

Definition 1.11. Let t be an $\mathcal{L}$-term. By induction on the complexity of t we define the set $FV(t)$ of *free variables* of t:

$$FV(x) = \{x\}, \quad FV(c) = \emptyset, \quad FV(f(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} FV(t_i)$$

for all $x \in \mathcal{V}$, all $c \in \mathcal{C}$ and all n-ary $f \in \mathcal{F}$.

Definition 1.12. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure. A map $e : \mathcal{V} \to A$ is called *evaluation of the variables of $\mathcal{L}$ in $\mathfrak{A}$*.

Definition 1.13. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, e an evaluation of the variables of $\mathcal{L}$ in $\mathfrak{A}$ and take an $\mathcal{L}$-term $t \in T_\mathcal{L}$. By induction on the complexity of t we define the *interpretation* of t in $\mathfrak{A}$ under e as

$$t^\mathfrak{A}[e] := \begin{cases} e(x) & \text{if } t = x \in \mathcal{V} \\ c^\mathfrak{A} & \text{if } t = c \in \mathcal{C} \\ f^\mathfrak{A}(t_1^\mathfrak{A}[e], \ldots, t_n^\mathfrak{A}[e]) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

If t is a constant term, then we will denote $t^\mathfrak{A}[e]$ by $t^\mathfrak{A}$.

Definition 1.14. The set $F_\mathcal{L}$ of *$\mathcal{L}$-formulae* is the smallest set such that

- if $t, s \in T_\mathcal{L}$, then $(t \doteq s) \in F_\mathcal{L}$,
- if $t_1, \ldots, t_n \in T_\mathcal{L}$ and $R \in \mathcal{R}$ is a relation symbol of arity $n \in \mathbb{N}$, then $R(t_1, \ldots, t_n) \in F_\mathcal{L}$,
- if $\varphi \in F_\mathcal{L}$, then $\neg\varphi \in F_\mathcal{L}$,
- if $\varphi, \psi \in F_\mathcal{L}$, then $\varphi \wedge \psi \in F_\mathcal{L}$,

- if $\varphi \in F_{\mathcal{L}}$ and $x \in \mathcal{V}$, then $\forall x\varphi \in F_{\mathcal{L}}$.

Formulas of the form $t \doteq s$ and $R(t_1, \ldots, t_n)$ are commonly called *atomic* formulas.

For $\varphi, \psi \in F_{\mathcal{L}}$ and $x \in \mathcal{V}$ we will use the following common abbreviations:

- $\varphi \vee \psi$ stands for $\neg(\neg\varphi \wedge \neg\psi)$,
- $\varphi \to \psi$ stands for $\neg\varphi \vee \psi$,
- $\varphi \leftrightarrow \psi$ stands for $(\varphi \to \psi) \wedge (\psi \to \varphi)$
- $\exists x\varphi$ stands for $\neg\forall x\neg\varphi$.

We omit the proof of the next result which is analogous to the proof of Lemma 1.9 above.

Lemma 1.15 (Induction on the complexity of formulas)**.** *Let P be a property of $\mathcal{L}$-formulas. Assume that $P(\varphi)$ holds for all atomic $\mathcal{L}$-formulas φ and that $P(\varphi)$ implies $P(\neg\varphi)$ and $P(\forall x\varphi)$ for all $\varphi \in F_{\mathcal{L}}$. Suppose in addition that $P(\varphi)$ and $P(\psi)$ imply $P(\varphi \wedge \psi)$ for all $\varphi, \psi \in F_{\mathcal{L}}$. Then $P(\varphi)$ holds for all $\varphi \in F_{\mathcal{L}}$.*

Definition 1.16. Let φ be an $\mathcal{L}$-formula. By induction on the complexity of φ we define the set $FV(\varphi)$ of *free variables* in φ:

- If φ is $t \doteq s$, then $FV(\varphi) = FV(t) \cup FV(s)$.
- If φ is $R(t_1, \ldots, t_n)$ for some $R \in \mathcal{R}$ of arity $n \in \mathbb{N}$, then

$$FV(\varphi) := \bigcup_{i=1}^{n} FV(t_i),$$

- If φ is $\neg\psi$, then $FV(\varphi) := FV(\psi)$,
- If φ is $\psi_1 \wedge \psi_2$, then $FV(\varphi) := FV(\psi_1) \cup FV(\psi_2)$,
- If φ is $\forall x\psi$, then $FV(\varphi) := FV(\psi) \setminus \{x\}$.

We call $\varphi \in F_{\mathcal{L}}$ an $\mathcal{L}$-*sentence* if $FV(\varphi) = \emptyset$.

Definition 1.17. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and e an evaluation of the variables in $\mathfrak{A}$. For $x \in \mathcal{V}$ and $a \in A$ we define another evaluation of the variables in $\mathfrak{A}$:

$$e(x/a)(y) := \begin{cases} a & \text{if } y = x \\ e(y) & \text{if } y \neq x \end{cases} \qquad (y \in \mathcal{V})$$

We now wish to tell what we mean by saying that a sentence is true in some structure.

Definition 1.18. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, e an evaluation of the variables of $\mathcal{L}$ and φ an $\mathcal{L}$-formula. We define "$\mathfrak{A} \models \varphi[e]$" by induction on the complexity of φ:

- if φ is $t \doteq s$, then $\mathfrak{A} \models \varphi[e]$ if and only if $t^{\mathfrak{A}}[e] = s^{\mathfrak{A}}[e]$ as elements of A.
- if φ is $R(t_1, \ldots, t_n)$ for some $R \in \mathcal{R}$ of arity $n \in \mathbb{N}$, then

$$\mathfrak{A} \models \varphi[e] \iff (t_1^{\mathfrak{A}}[e], \ldots, t_n^{\mathfrak{A}}[e]) \in R^{\mathfrak{A}};$$

- If φ is $\neg\psi$, then $\mathfrak{A} \models \varphi[e]$ if and only if $\mathfrak{A} \models \psi[e]$ does not hold.
- If φ is $\psi_1 \wedge \psi_2$, then $\mathfrak{A} \models \varphi[e]$ if and only if $\mathfrak{A} \models \psi_i[e]$ for $i = 1, 2$.
- If φ is $\forall x\psi$, then $\mathfrak{A} \models \varphi[e]$ if and only if $\mathfrak{A} \models \psi[e(x/a)]$ for all $a \in A$.

If $\mathfrak{A} \models \varphi[e]$, then we say that φ is *true in* $\mathfrak{A}$ *under* e or that $\mathfrak{A}$ *satisfies* φ *under* e.

We write $\mathfrak{A} \models \varphi$ if $\mathfrak{A} \models \varphi[e]$ for all evaluations of the variables e. If $\mathfrak{A} \models \varphi$, then we say that φ is *true in* $\mathfrak{A}$ or that $\mathfrak{A}$ is a *model of* φ or that φ *holds in* $\mathfrak{A}$.

Remark 1.19. For an $\mathcal{L}$-formula φ with $FV(\varphi) = \{x_1, \ldots, x_n\}$ we denote its *universal closure*, i.e., the $\mathcal{L}$-sentence $\forall x_1 \ldots \forall x_n \varphi$ by $\forall\varphi$. Let φ be an arbitrary $\mathcal{L}$-formula and $\mathfrak{A}$ an $\mathcal{L}$-structure. Then, by the above definition, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \models \forall\varphi$.

Observe further that if φ is an $\mathcal{L}$-sentence, then $\mathfrak{A} \models \varphi[e]$ for some evaluation of the variables e if and only if $\mathfrak{A} \models \varphi$.

1.3 Theories and elementary equivalence

For the whole section we fix a language $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$. Any set $\mathbf{T}$ of $\mathcal{L}$-sentences will be called an $\mathcal{L}$-*theory*. In the literature, there are also other definitions of this notion, but the additional properties required for theories are not of too much interest for this work.

Theories usually appear in two different forms. First, we may have a (possibly infinite) list of $\mathcal{L}$-sentences. Such lists are usually called *axiom systems* and are $\mathcal{L}$-theories in our sense. Second, the set of $\mathcal{L}$-sentences which are true simultaneously in all members of a given class of $\mathcal{L}$-structures is an $\mathcal{L}$-theory. An example for this is the set of all $\mathcal{L}_f$-sentences which are true in every finite field.

Given a single $\mathcal{L}$-structure $\mathfrak{A}$, the set of all $\mathcal{L}$-sentences which are true in $\mathfrak{A}$ is called the *theory of* $\mathfrak{A}$ and denoted by $\mathrm{Th}(\mathfrak{A})$.

Definition 1.20. We say that two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* and write $\mathfrak{A} \equiv \mathfrak{B}$ if for all $\mathcal{L}$-sentences φ we have that $\mathfrak{A}$ is a model of φ if and only if $\mathfrak{B}$ is a model of φ. In symbols,

$$\mathfrak{A} \equiv \mathfrak{B} \iff \mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B}).$$

Let us now prove that isomorphic structures are elementarily equivalent.

Proposition 1.21. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures and $\sigma : A \to B$ an isomorphism. Then for all $\varphi \in F_{\mathcal{L}}$ and for all evaluations of the variables e we have that*

$$\mathfrak{A} \models \varphi[e] \iff \mathfrak{B} \models \varphi[\sigma \circ e].$$

In particular, $\mathfrak{A} \simeq \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}$.

Proof. Let t be an $\mathcal{L}$-term. By induction on the complexity of t it is easy to show that

$$\sigma(t^{\mathfrak{A}}[e]) = t^{\mathfrak{B}}[\sigma \circ e]. \tag{1.1}$$

Fix an $\mathcal{L}$-formula φ. We proceed by induction on the complexity of φ.

If φ is $t \doteq s$, then, using the bijectivity of σ, we obtain

$$\begin{aligned} \mathfrak{A} \models \varphi[e] &\iff t^{\mathfrak{A}}[e] = s^{\mathfrak{A}}[e] \\ &\iff \sigma(t^{\mathfrak{A}}[e]) = \sigma(s^{\mathfrak{A}}[e]) \\ &\iff t^{\mathfrak{B}}[\sigma \circ e] = s^{\mathfrak{B}}[\sigma \circ e] \\ &\iff \mathfrak{B} \models \varphi[\sigma \circ e]. \end{aligned}$$

If φ is $R(t_1, \ldots, t_n)$ for some n-ary $R \in \mathcal{R}$, then, using the fact that σ is a strict morphism, we obtain

$$\begin{aligned} \mathfrak{A} \models \varphi[e] &\iff (t_1^{\mathfrak{A}}[e], \ldots, t_n^{\mathfrak{A}}[e]) \in R^{\mathfrak{A}} \\ &\iff (\sigma t_1^{\mathfrak{A}}[e], \ldots, \sigma t_n^{\mathfrak{A}}[e]) \in R^{\mathfrak{B}} \\ &\iff (t_1^{\mathfrak{B}}[\sigma \circ e], \ldots, t_n^{\mathfrak{B}}[\sigma \circ e]) \in R^{\mathfrak{B}} \\ &\iff \mathfrak{B} \models \varphi[\sigma \circ e]. \end{aligned}$$

If φ is $\neg\psi$, then, using the induction hypothesis, we obtain

$$\begin{aligned} \mathfrak{A} \models \varphi[e] &\iff \mathfrak{A} \not\models \psi[e] \\ &\iff \mathfrak{B} \not\models \psi[\sigma \circ e] \\ &\iff \mathfrak{B} \models \varphi[\sigma \circ e]. \end{aligned}$$

If φ is $\psi_1 \wedge \psi_2$, then, using the induction hypothesis, we obtain

$$\begin{aligned} \mathfrak{A} \models \varphi[e] &\iff \mathfrak{A} \models \psi_i[e] \ (i = 1, 2) \\ &\iff \mathfrak{B} \models \psi_i[\sigma \circ e] \ (i = 1, 2) \\ &\iff \mathfrak{B} \models \varphi[\sigma \circ e]. \end{aligned}$$

If φ is $\forall x \psi$, then since σ is surjective and using the induction hypothesis, we obtain

$$\begin{aligned} \mathfrak{A} \models \varphi[e] &\iff \mathfrak{A} \models \psi[e(x/a)] \text{ for all } a \in A \\ &\iff \mathfrak{B} \models \psi[\sigma \circ e(x/a)] \text{ for all } a \in A \\ &\iff \mathfrak{B} \models \psi[(\sigma \circ e)(x/\sigma a)] \text{ for all } a \in A \\ &\iff \mathfrak{B} \models \psi[(\sigma \circ e)(x/b)] \text{ for all } b \in B \\ &\iff \mathfrak{B} \models \varphi[\sigma \circ e]. \end{aligned}$$

The last assertion follows from Remark 1.19. □

Definition 1.22. Let $\mathfrak{B}$ be an $\mathcal{L}$-structure. A substructure $\mathfrak{A}$ of $\mathfrak{B}$ is called *elementary substructure* if for all $\mathcal{L}$-formulae φ and for all evaluations of the variables e in $\mathfrak{A}$, we have

$$\mathfrak{A} \models \varphi[e] \iff \mathfrak{B} \models \varphi[e].$$

We write $\mathfrak{A} \preceq \mathfrak{B}$ if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$.

It is clear that $\mathfrak{A} \preceq \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}$. The converse does not hold, as the following example shows. Let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. Then $(\mathbb{N}^+, \leq)$ is a substructure of $(\mathbb{N}, \leq)$. In addition, we have $(\mathbb{N}^+, \leq) \simeq (\mathbb{N}, \leq)$ with the isomorphism given by $a \mapsto a-1$. Therefore, Proposition 1.21 yields $(\mathbb{N}^+, \leq) \equiv (\mathbb{N}, \leq)$. However, $(\mathbb{N}^+, \leq)$ is not an elementary substructure of $(\mathbb{N}, \leq)$. To see this let e be any evaluation with $e(x) = 1$, the formula $\forall y (x \leq y)$ is true in $(\mathbb{N}^+, \leq)$ under e but it is not true in $(\mathbb{N}, \leq)$ under e.

The following is an easy and yet useful characterization of elementary substructures.

Lemma 1.23. *For $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ we have that $\mathfrak{A} \preceq \mathfrak{B}$ if and only if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and $(\mathfrak{A}, A) \equiv (\mathfrak{B}, A)$.*

Proof. If $\mathfrak{A} \preceq \mathfrak{B}$, then $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. Take an $\mathcal{L}(A)$-sentence φ. By replacing the finitely many constant symbols c_a for $a \in A$ which occur in φ by variables x_a of $\mathcal{L}$ we obtain an $\mathcal{L}$-formula φ'. Let e be an evaluation of the variables such that $e(x_a) = a$ for all $a \in A$ such that c_a occurs in φ. Since $\mathfrak{A} \preceq \mathfrak{B}$ we obtain

$$(\mathfrak{A}, A) \models \varphi \iff \mathfrak{A} \models \varphi'[e] \iff \mathfrak{B} \models \varphi'[e] \iff (\mathfrak{B}, A) \models \varphi.$$

Thus, $(\mathfrak{A}, A) \equiv (\mathfrak{B}, A)$.

For the converse, note that for any evaluation e of the variables of $\mathcal{L}$ in $\mathfrak{A}$ we can obtain, from an $\mathcal{L}$-formula φ, a $\mathcal{L}(A)$-sentence by replacing a free variable x of φ by $c_{e(x)}$. We then have

$$\mathfrak{A} \models \varphi[e] \Longleftrightarrow (\mathfrak{A}, A) \models \varphi' \Longleftrightarrow (\mathfrak{B}, A) \models \varphi' \Longleftrightarrow \mathfrak{B} \models \varphi[e]. \qquad \square$$

Definition 1.24. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. An embedding $\sigma : A \to B$ is called *elementary embedding* if for every $\mathcal{L}$-formula φ and every evaluation $e : \mathcal{V} \to A$, we have

$$\mathfrak{A} \models \varphi[e] \iff \mathfrak{B} \models \varphi[\sigma \circ e].$$

Note that $\sigma : A \to B$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$ if and only if the image of $\mathfrak{A}$ in $\mathfrak{B}$ is an elementary substructure of $\mathfrak{B}$.

Let $\mathbf{T}$ be a non-empty $\mathcal{L}$-theory. We say that an $\mathcal{L}$-structure $\mathfrak{A}$ is a *model* of $\mathbf{T}$ if $\mathfrak{A} \models \psi$ for all $\psi \in \mathbf{T}$. Let φ be an $\mathcal{L}$-formula. Then we write $\mathbf{T} \models \varphi$ if $\mathfrak{A} \models \varphi$ for every model $\mathfrak{A}$ of $\mathbf{T}$. To conclude this section we recall the important Finiteness Theorem. For a proof we refer, for example, to [43, Theorem 1.5.6].

Theorem 1.25. *Take a non-empty $\mathcal{L}$-theory $\mathbf{T}$. If φ is an $\mathcal{L}$-sentence such that $\mathbf{T} \models \varphi$, then there are $\psi_1, \ldots, \psi_n \in \mathbf{T}$ such that $\{\psi_1, \ldots, \psi_n\} \models \varphi$.*

1.4 Quantifier elimination

If an $\mathcal{L}$-formula φ holds in every $\mathcal{L}$-structure we will write $\models \varphi$ and we will say that φ and ψ are *equivalent* $\mathcal{L}$-formulae if $\models \varphi \leftrightarrow \psi$. If $\mathbf{T}$ is an $\mathcal{L}$-theory, then we say that two $\mathcal{L}$-formulae φ and ψ are $\mathbf{T}$-*equivalent* if $\mathbf{T} \models \varphi \leftrightarrow \psi$.

An $\mathcal{L}$-formula φ is called *quantifier free* if no quantifier appears in it. An $\mathcal{L}$-theory $\mathbf{T}$ is said to *admit quantifier elimination* if every $\mathcal{L}$-formula is $\mathbf{T}$-equivalent to a quantifier free $\mathcal{L}$-formula.

There is a slight difficulty that we should discuss here. If φ is an $\mathcal{L}$-sentence such that $\mathbf{T} \models \varphi$ or $\mathbf{T} \models \neg\varphi$, then it must be $\mathbf{T}$-equivalent to a quantifier free $\mathcal{L}$-sentence which holds in every or in no model of $\mathbf{T}$, respectively. However, if $\mathcal{L}$ contains no constant symbol, then there are no quantifier free $\mathcal{L}$-sentences at all. Having this case in mind, we add two atomic sentences to the set of all $\mathcal{L}$-sentences: $\top$ called *top*, and $\perp$ called *bottom*. These are quantifier free sentences, but the interpretation of $\top$ is defined to be the same as that of the sentence $\forall x(x \doteq x)$, which holds in all $\mathcal{L}$-structures, and the interpretation of $\perp$ in an $\mathcal{L}$-structure is defined to be the same as that of the sentence $\exists x \neg(x \doteq x)$, which does not hold in any $\mathcal{L}$-structure.

Let $\mathfrak{S}$ be a substructure of an $\mathcal{L}$-structure $\mathfrak{A}$ with universe S. Since $S \subseteq A$, $\mathfrak{A}$ is expanded in a canonical way to an $\mathcal{L}(S)$-structure $(\mathfrak{A}, S)$. Then $(\mathfrak{S}, S)$ is a substructure of $(\mathfrak{A}, S)$. Now let $\mathfrak{S}$ be a common substructure of two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$. We will say that $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent over* $\mathfrak{S}$ and write $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$ if $(\mathfrak{A}, S) \equiv (\mathfrak{B}, S)$. The constants inferred from a substructure are often called *parameters*. For example if $\mathfrak{A} \preceq \mathfrak{B}$, then $\mathfrak{A} \equiv_{\mathfrak{A}} \mathfrak{B}$ as we have noted in Lemma 1.23. Note that $\equiv_{\mathfrak{S}}$ is an equivalence relation.

Let $\mathbf{S}$ be a set of $\mathcal{L}$-sentences and $\mathfrak{A}$, $\mathfrak{B}$ two $\mathcal{L}$-structures. We will write

$$\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$$

if for all $\varphi \in \mathbf{S}$ we have that $\mathfrak{A} \models \varphi$ implies that $\mathfrak{B} \models \varphi$. For example, $\mathfrak{A} \equiv \mathfrak{B}$ means $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ and $\mathfrak{B} \Rightarrow_{\mathbf{S}} \mathfrak{A}$ for $\mathbf{S}$ the set of all $\mathcal{L}$-sentences. If $\mathbf{S} = \{\varphi\}$ the we will write $\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B}$ in place of $\mathfrak{A} \Rightarrow_{\{\varphi\}} \mathfrak{B}$. Observe that $\Rightarrow_{\mathbf{S}}$ is transitive.

Every $\mathcal{L}$-structure is either a model of φ or of $\neg\varphi$. That is, $\mathfrak{A} \not\models \varphi$ is equivalent to $\mathfrak{A} \models \neg\varphi$. By the principle of contraposition, $\mathfrak{A} \models \varphi$ implies $\mathfrak{B} \models \varphi$ is equivalent to $\mathfrak{B} \not\models \varphi$ implies $\mathfrak{A} \not\models \varphi$, which in turn is equivalent to $\mathfrak{B} \models \neg\varphi$ implies $\mathfrak{A} \models \neg\varphi$. This proves that

$$\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B} \text{ if and only if } \mathfrak{B} \Rightarrow_{\neg\varphi} \mathfrak{A}.$$

If we set $\neg\mathbf{S} := \{\neg\varphi \mid \varphi \in \mathbf{S}\}$, then we obtain

$$\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B} \text{ if and only if } \mathfrak{B} \Rightarrow_{\neg\mathbf{S}} \mathfrak{A}.$$

Since if φ and ψ are equivalent $\mathcal{L}$-formulae, then $\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B}$ if and only if $\mathfrak{A} \Rightarrow_{\psi} \mathfrak{B}$, and since φ is equivalent to $\neg\neg\varphi$, we deduce the following result.

Lemma 1.26. *If* $\mathbf{S}$ *is closed under negation, then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\mathfrak{B} \Rightarrow_{\mathbf{S}} \mathfrak{A}$.

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and A_0 be the subset of A which consists of the interpretations of all constant $\mathcal{L}$-terms in $\mathfrak{A}$. Since the set $T_{\mathcal{L}}^c$ is the closure of the set $\mathcal{C}$ under application of the function symbols $f \in \mathcal{F}$, the set A_0 is the closure in A of the set of all interpretations of constant symbols under the application of the interpretations of function symbols. Consequently, if we interpret the functions and relations of $\mathcal{L}$ by the restrictions to A_0 of their interpretations on A, then we obtain a substructure $\mathfrak{A}_0$ of $\mathfrak{A}$ with universe A_0. It is uniquely determined by $\mathfrak{A}$. Let us call it the *constant substructure of* $\mathfrak{A}$. Every substructure of $\mathfrak{A}$ will also contain $\mathfrak{A}_0$. Note that $\mathfrak{A}$ satisfies the same atomic $\mathcal{L}$-sentences as its constant substructure.

Example 1.27. The constant substructure of a field K in the language of fields is its prime field which is $\mathbb{Q}$ if K has characteristic 0 and is $\mathbb{F}_p$ if K has characteristic $p > 0$. Note that, if K has characteristic 0, then its constant substructure in the language of rings is $\mathbb{Z}$.

Let now $\mathfrak{B}$ be a second $\mathcal{L}$-structure, with constant substructure $\mathfrak{B}_0$. We introduce an equivalence relation between A_0 and B_0 as follows. For all $a \in A_0$ and $b \in B_0$ we write $\mathrm{ctr}(a, b)$ if and only if there is a constant $\mathcal{L}$-term t such that $t^{\mathfrak{A}} = a$ and $t^{\mathfrak{B}} = b$. We call this relation the *constant term relation* between $\mathfrak{A}$ and $\mathfrak{B}$.

Lemma 1.28. *1. Let* $\mathbf{S}$ *be the set of all atomic sentences* $t_1 \doteq t_2$ *with* $t_1, t_2 \in T^c_{\mathcal{L}}$*. Then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ *is a map from* A_0 *to* B_0*.*

2. *Let* $\mathbf{S}$ *be the set of all atomic* $\mathcal{L}$*-sentences. Then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ *is a morphism from* $\mathfrak{A}_0$ *to* $\mathfrak{B}_0$*.*

3. *Let* $\mathbf{S}$ *be the set of all atomic* $\mathcal{L}$*-sentences and their negations, or of all quantifier free* $\mathcal{L}$*-sentences. Then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ *is an isomorphism from* $\mathfrak{A}_0$ *to* $\mathfrak{B}_0$*.*

4. *Let* $\mathbf{S}$ *be the set of all* $\mathcal{L}$*-sentences and assume that* $\mathfrak{A}_0 = \mathfrak{A}$*. Then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ *is an elementary embedding from* $\mathfrak{A}$ *to* $\mathfrak{B}$*.*

Proof.

1. Let $a_1, a_2 \in A_0$ and $t_1, t_2 \in T^c_{\mathcal{L}}$ such that $t_1^{\mathfrak{A}} = a_1$ and $t_2^{\mathfrak{A}} = a_2$. Let b_1 and b_2 be the interpretations of t_1 and t_2 in B. Then $a_1 = a_2$ implies $b_1 = b_2$ if and only if $\mathfrak{A} \models t_1 \doteq t_2$ implies $\mathfrak{B} \models t_1 \doteq t_2$. This shows that $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ is a map if and only if $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$.

2. If $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ is a morphism from $\mathfrak{A}_0$ to $\mathfrak{B}_0$, then in view of Part 1. it remains to show that $\mathfrak{A}_0 \models R(t_1, \ldots, t_n)$ implies $\mathfrak{B}_0 \models R(t_1, \ldots, t_n)$ for every $R \in \mathcal{R}$ of arity n. However, this is just property (M1) of a morphism. Conversely, if $\mathfrak{A} \models \varphi$ implies $\mathfrak{B} \models \varphi$ for every atomic $\mathcal{L}$-sentence φ, then we obtain that $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ satisfies property (M1) and by part 1. that it is a map from A_0 to B_0. By its definition it satisfies property (M3) and it follows from the rules for the interpretation of terms that it satisfies property (M2).

3. If $\mathbf{S}$ is the set of all atomic $\mathcal{L}$-sentences and their negations, then the proof is similar to that of Part 2. so we have to deal with injectivity and (M1'). However, these follow from Lemma 1.26. For the case of $\mathbf{S}$ being the set of all quantifier free $\mathcal{L}$-sentences, we only have to note the following: the set of all quantifier free $\mathcal{L}$-sentences holding in an $\mathcal{L}$-structure $\mathfrak{A}$ is uniquely determined by the set of all atomic $\mathcal{L}$-sentences holding in $\mathfrak{A}$.

4. Our hypothesis $\mathfrak{A}_0 = \mathfrak{A}$ means that for each $a \in A$ there is a constant $\mathcal{L}$-term whose interpretation in $\mathfrak{A}$ is a. From this we derive that every $\mathcal{L}(A)$-sentence φ is equivalent to an $\mathcal{L}$-sentence obtained from φ through replacing the

constant symbol c_a by some $\mathcal{L}$-term whose interpretation in $\mathfrak{A}$ is a. Therefore, $\mathfrak{B}_0 \preceq \mathfrak{B}$ if and only if $\mathfrak{B}_0 \Rightarrow_{\mathbf{S}} \mathfrak{B}$.

By Proposition 1.21, if $\mathfrak{A} \simeq \mathfrak{B}_0$, then $\mathfrak{B}_0 \Rightarrow_{\mathbf{S}} \mathfrak{A}$ and $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}_0$. Hence if $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$, then by Part 3. $\mathfrak{A} \simeq \mathfrak{B}_0$ and thus $\mathfrak{B}_0 \Rightarrow_{\mathbf{S}} \mathfrak{A}$; by transitivity, it follows that $\mathfrak{B}_0 \Rightarrow_{\mathbf{S}} \mathfrak{B}$ and therefore $\mathfrak{B}_0 \preceq \mathfrak{B}$.

For the converse, assume that $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ is an elementary embedding of $\mathfrak{A}_0$ in $\mathfrak{B}$. That is, $\mathfrak{A} \simeq \mathfrak{B}_0$ and $\mathfrak{B}_0 \preceq \mathfrak{B}$ (note that here we used Lemma 1.23). The former implies that $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}_0$ and the latter implies that $\mathfrak{B}_0 \Rightarrow_{\mathbf{S}} \mathfrak{B}$. By transitivity this gives $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$. □

Corollary 1.29. *For arbitrary $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, the following assertions are equivalent:*

(*i*) *$\mathfrak{A}$ and $\mathfrak{B}$ have a common substructure up to isomorphism,*

(*ii*) *$\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ with $\mathbf{S}$ the set of all quantifier free $\mathcal{L}$-sentences,*

(*iii*) *$\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same atomic $\mathcal{L}$-sentences.*

Proof. Since (1.1) follows from properties (M2) and (M3), it follows that for every constant $\mathcal{L}$-term t we have that $\sigma t^{\mathfrak{A}} = t^{\mathfrak{B}}$ for any morphism σ. Thus, the restriction to $\mathfrak{A}_0$ of every morphism from some substructure $\mathfrak{A}_1$ of $\mathfrak{A}$ into $\mathfrak{B}$ must coincide with $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$. Hence if a substructure $\mathfrak{A}_1$ of $\mathfrak{A}$ is isomorphic to a substructure $\mathfrak{B}_1$ of $\mathfrak{B}$, then the restriction of this isomorphism will be an isomorphism of $\mathfrak{A}_0$ onto $\mathfrak{B}_0$. Therefore, (*i*) ⇔ (*ii*) follows from Part 3. of the foregoing lemma.

For the equivalence (*ii*) ⇔ (*iii*), note that, by Lemma 1.26, $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ implies $\mathfrak{B} \Rightarrow_{\mathbf{S}} \mathfrak{A}$, and that, as pointed out earlier, the set of quantifier free $\mathcal{L}$-sentences holding in an $\mathcal{L}$-structure $\mathfrak{A}$ is uniquely determined by the set of all atomic $\mathcal{L}$-sentences holding in $\mathfrak{A}$. □

Definition 1.30. An $\mathcal{L}$-theory $\mathbf{T}$ is called *substructure complete* if $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$ for every two models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathbf{T}$ and every common substructure $\mathfrak{S}$ of $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 1.31. *A theory $\mathbf{T}$ admits quantifier elimination if and only if it is substructure complete.*

We need some preparations for the proof of this theorem. Take any set $\mathbf{S}$ of $\mathcal{L}$-formulas. Let $\mathbf{S}^{\vee}$ denote the set $\{\psi_1 \vee \ldots \vee \psi_n \mid n \in \mathbb{N}, \psi_i \in \mathbf{S}\}$ of all finite disjunctions of formulas of $\mathbf{S}$. We take $\perp$ to be the empty disjunction and include it in $\mathbf{S}^{\vee}$. Similarly, we define $\mathbf{S}^{\wedge}$, which is the closure of $\mathbf{S}$ under taking finite conjunctions. We take $\top$ to be the empty conjunction and include it in $\mathbf{S}^{\wedge}$. We write $\mathbf{S}^{\vee\wedge}$ for $(\mathbf{S}^{\vee})^{\wedge}$. Observe that $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ implies that $\mathfrak{A} \Rightarrow_{\mathbf{S}^{\vee\wedge}} \mathfrak{B}$.

Lemma 1.32. *Let* $\mathbf{T}$ *be an* $\mathcal{L}$*-theory and* $\mathbf{S}$ *any set of* $\mathcal{L}$*-sentences. Then the following are equivalent for every* $\mathcal{L}$*-sentence* φ*.*

(*i*) φ *is* $\mathbf{T}$*-equivalent to a sentence in* $\mathbf{S}^{\vee\wedge}$*,*

(*ii*) *for all* $\mathfrak{A}$*,* $\mathfrak{B}$ *models of* $\mathbf{T}$ *such that* $\mathfrak{A} \Rrightarrow_{\mathbf{S}} \mathfrak{B}$*, the sentence* φ *satisfies* $\mathfrak{A} \Rrightarrow_{\varphi} \mathfrak{B}$*.*

Proof. Assume (*i*) and that $\mathfrak{A}$, $\mathfrak{B}$ are models of $\mathbf{T}$. If $\mathfrak{A} \Rrightarrow_{\mathbf{S}} \mathfrak{B}$ and thus also $\mathfrak{A} \Rrightarrow_{\mathbf{S}^{\vee\wedge}} \mathfrak{B}$ holds and if φ is $\mathbf{T}$-equivalent to a sentence in $\mathbf{S}^{\vee\wedge}$, then also $\mathfrak{A} \Rrightarrow_{\varphi} \mathfrak{B}$ holds.

Now assume (*ii*). We may also assume that $\mathbf{T} \vDash \varphi$ since otherwise, φ is $\mathbf{T}$-equivalent to $\bot$. Set $\mathbf{S}_\varphi := \{\psi \in \mathbf{S}^\vee \mid \mathbf{T} \vDash \varphi \to \psi\}$. Suppose that $\mathbf{T} \cup \mathbf{S}_\varphi \nvDash \varphi$. By definition, this means that there is a model $\mathfrak{B}$ of $\mathbf{T} \cup \mathbf{S}_\varphi$ such that $\mathfrak{B} \vDash \neg\varphi$. We set $\mathbf{T}' := \{\neg\psi \mid \psi \in \mathbf{S} \text{ and } \mathfrak{B} \vDash \neg\psi\}$. With this definition, for every model $\mathfrak{A}$ of $\mathbf{T} \cup \mathbf{T}'$ we obtain that $\mathfrak{A} \Rrightarrow_{\mathbf{S}} \mathfrak{B}$; because of $\mathfrak{B} \vDash \neg\varphi$ and (*ii*), this implies that $\mathfrak{A} \vDash \neg\varphi$. We conclude that $\mathbf{T} \cup \mathbf{T}' \vDash \neg\varphi$. As $\mathbf{T} \vDash \neg\varphi$ is excluded by assumption, $\mathbf{T}'$ cannot be empty. By the Finiteness Theorem 1.25, there are $\neg\psi_1, \ldots, \neg\psi_n \in \mathbf{T}'$ such that $\mathbf{T} \cup \{\psi_1, \ldots, \psi_n\} \vDash \neg\varphi$. It follows that in every model of $\mathbf{T}$ in which φ holds, at least one of the $\neg\psi_i$ does not hold. In other words, $\mathbf{T} \vDash \varphi \to (\psi_1 \vee \ldots \vee \psi_n)$. Since $\psi_1 \vee \ldots \vee \psi_n \in \mathbf{S}^\vee$, it follows that $\psi_1 \vee \ldots \vee \psi_n \in \mathbf{S}_\varphi$. Since $\mathfrak{B} \vDash \mathbf{S}_\varphi$, we conclude that $\mathfrak{B} \vDash \psi_1 \vee \ldots \vee \psi_n$. Nevertheless, by our choice of the ψ_i, we have that $\mathfrak{B} \vDash \neg\psi_i$ for all i. This contradiction shows that $\mathbf{T} \cup \mathbf{S}_\varphi \vDash \varphi$.

If $\mathbf{S}_\varphi = \emptyset$, then we find that $\mathbf{T} \vDash \varphi$, in which case φ is $\mathbf{T}$-equivalent to $\top$. If $\mathbf{S}_\varphi \neq \emptyset$ then again by the Finiteness Theorem 1.25, there are $\psi_1, \ldots, \psi_n \in \mathbf{S}_\varphi$ such that $\mathbf{T} \cup \{\psi_1, \ldots, \psi_n\} \vDash \varphi$. In other words, $\mathbf{T} \vDash \psi_1 \wedge \ldots \wedge \psi_n \to \varphi$. However, by definition of $\mathbf{S}_\varphi$, we have that $\mathbf{T} \vDash \varphi \to \psi_i$ for all i. Hence, $\mathbf{T} \vDash \varphi \leftrightarrow \psi$ for $\psi := \psi_1 \wedge \ldots \wedge \psi_n \in \mathbf{S}_\varphi^\wedge \subseteq \mathbf{S}^{\vee\wedge}$. □

In what follows, for an $\mathcal{L}$-formula φ we write $\varphi(x_1, \ldots, x_n)$ to indicate that $FV(\varphi) = \{x_1, \ldots, x_n\}$. If $\mathfrak{A}$ is an $\mathcal{L}$-structure and $a_1, \ldots, a_n \in A$, then, for $A' \subseteq A$ such that $a_1, \ldots, a_n \in A'$, we write $(\mathfrak{A}, A') \vDash \varphi(a_1, \ldots, a_n)$ to indicate that the free variables of φ have been substituted by the constant symbols $c_{a_1}, \ldots, c_{a_n}$ of $\mathcal{L}(A')$ to obtain an $\mathcal{L}(A')$-sentence which holds in $(\mathfrak{A}, A')$.

Let $\mathbf{T}$ be an $\mathcal{L}$-theory and $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula. When we extend $\mathcal{L}$ to $\mathcal{L}(x_1, \ldots, x_n) := \mathcal{L}(\{x_1, \ldots, x_n\})$, then $\varphi(x_1, \ldots, x_n)$ becomes an $\mathcal{L}(x_1, \ldots, x_n)$-sentence. Every $\mathcal{L}$-sentence φ is also an $\mathcal{L}(x_1, \ldots, x_n)$-sentence (here we assume the variables which are not free in φ to be different from $x_1, \ldots, x_n$). Hence $\mathbf{T}$ is also an $\mathcal{L}(x_1, \ldots, x_n)$-theory. The only point is that we have to show that the $\mathcal{L}$-theory $\mathbf{T}$ satisfies $\mathbf{T} \vDash \forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$ if and only if the $\mathcal{L}(x_1, \ldots, x_n)$-theory $\mathbf{T}$ satisfies $\mathbf{T} \vDash \varphi$. However, every $\mathcal{L}(x_1, \ldots, x_n)$-structure arises from an $\mathcal{L}$-structure by expansion. Given an $\mathcal{L}$-structure $\mathfrak{A}$, it becomes an $\mathcal{L}(x_1, \ldots, x_n)$-structure by an arbitrary choice of the interpretation for the new constant symbols $x_1, \ldots, x_n$.

That is, for every n-tuple $a_1, \dots, a_n$, there is an expansion of $\mathfrak{A}$ in which x_i is interpreted by a_i, for each i. Consequentely, $\mathfrak{A} \models \forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n)$ if and only if every expansion of $\mathfrak{A}$ to an $\mathcal{L}(x_1, \dots, x_n)$-structure is a model of φ. This proves what we wanted. We will use this approach in the proof of the following lemma.

We will further use a principle which is commonly used in algebra. Assume that two structures $\mathfrak{A}$ and $\mathfrak{B}$ have substructures $\mathfrak{A}_0$ and $\mathfrak{B}_0$ which admit an isomorphism $\sigma : A_0 \to B_0$. We then would like to identify $\mathfrak{A}_0$ with $\mathfrak{B}_0$ and thus assume that $\mathfrak{A}$ and $\mathfrak{B}$ have a common substructure. To obtain this, one extends the isomorphism σ from $\mathfrak{A}_0$ to an isomorphism of $\mathfrak{A}$ onto some structure $\mathfrak{A}'$. Now $\mathfrak{A}'$ and $\mathfrak{B}$ indeed have $\mathfrak{B}_0$ as a common substructure. Similarly, one can also find an isomorphic image $\mathfrak{B}'$ of $\mathfrak{B}$ such that $\mathfrak{A}$ and $\mathfrak{B}'$ have $\mathfrak{A}_0$ as a common substructure.

Lemma 1.33. *Let* $\mathbf{T}$ *be an* $\mathcal{L}$*-theory. Then the following are equivalent for every* $\mathcal{L}$*-formula* $\varphi(x_1, \dots, x_n)$*:*

(*i*) $\varphi(x_1, \dots, x_n)$ *is* $\mathbf{T}$*-equivalent to a quantifier-free* $\mathcal{L}$*-formula* $\psi(x_1, \dots, x_n)$*,*

(*ii*) *for all* $\mathfrak{A}$*,* $\mathfrak{B}$ *models of* $\mathbf{T}$*, every common substructure* $\mathfrak{S}$ *of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *and for all* n*-tuples* $(a_1, \dots, a_n) \in S^n$*,*

$$(\mathfrak{A}, S) \models \varphi(a_1, \dots, a_n) \Longrightarrow (\mathfrak{B}, S) \models \varphi(a_1, \dots, a_n).$$

Proof. Assume (*i*) and let $\mathfrak{A}$ and $\mathfrak{B}$ be models of $\mathbf{T}$ with a common substructure $\mathfrak{S}$. Pick $(a_1, \dots, a_n) \in S^n$. If $\psi(x_1, \dots, x_n)$ is a quantifier-free $\mathcal{L}$-formula, then $\psi(a_1, \dots, a_n)$ is a quantifier-free $\mathcal{L}(S)$-sentence. Hence by Corollary 1.29, $(\mathfrak{A}, S) \models \psi(a_1, \dots, a_n)$ implies $(\mathfrak{B}, S) \models \psi(a_1, \dots, a_n)$. Therefore, if $\mathbf{T} \models \varphi \leftrightarrow \psi$, then $(\mathfrak{A}, S) \models \varphi(a_1, \dots, a_n) \leftrightarrow \psi(a_1, \dots, a_n)$ and $(\mathfrak{B}, S) \models \varphi(a_1, \dots, a_n) \leftrightarrow \psi(a_1, \dots, a_n)$. Thus, $(\mathfrak{A}, S) \models \varphi(a_1, \dots, a_n)$ implies $(\mathfrak{B}, S) \models \varphi(a_1, \dots, a_n)$ and (*ii*) follows.

Now assume (*ii*). By viewing $x_1, \dots, x_n$ as new constant symbols, we have that $\varphi(x_1, \dots, x_n)$ is an $\mathcal{L}(x_1, \dots, x_n)$-sentence. Let $\mathbf{S}$ be the set of all quantifier-free $\mathcal{L}(x_1, \dots, x_n)$-sentences, and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}(x_1, \dots, x_n)$-structures. Then by Corollary 1.29 $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ implies that $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic substructures. Identifying them we can assume that $\mathfrak{A}$ and $\mathfrak{B}$ have a common substructure $\mathfrak{S}$. Therefore, by (*ii*) we have that

$$(\mathfrak{A}, S) \models \varphi(a_1, \dots, a_n) \Longrightarrow (\mathfrak{B}, S) \models \varphi(a_1, \dots, a_n).$$

Now we can apply Lemma 1.32 to deduce that $\varphi(x_1, \dots, x_n)$ is $\mathbf{T}$-equivalent to some $\psi(x_1, \dots, x_n) \in \mathbf{S}^{\vee\wedge} = \mathbf{S}$. □

Now we can give the proof of Theorem 1.31.

Proof of Theorem 1.31. Assume that $\mathbf{T}$ admits quantifier elimination, and let $\mathfrak{A}$ and $\mathfrak{B}$ be models of $\mathbf{T}$ with a common substructure $\mathfrak{S}$. We have to show that $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$ or equivalently, that $(\mathfrak{A}, S) \Rightarrow_{\varphi} (\mathfrak{B}, S)$ for every $\mathcal{L}(S)$-sentence φ. We can write φ as $\varphi(a_1, \ldots, a_n)$ with $\varphi(x_1, \ldots, x_n)$ an $\mathcal{L}$-formula. Since $\mathbf{T}$ admits quantifier elimination, the previous lemma shows that

$$(\mathfrak{A}, S) \vDash \varphi(a_1, \ldots, a_n) \Longrightarrow (\mathfrak{B}, S) \vDash \varphi(a_1, \ldots, a_n),$$

as required.

For the converse, assume that $\mathbf{T}$ is substructure complete. Then condition (*ii*) of Lemma 1.33 holds for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ because $\varphi(a_1, \ldots, a_n)$ is an $\mathcal{L}(S)$-sentence. Hence condition (*i*) of Lemma 1.33 holds for every $\mathcal{L}$-formula, i.e., $\mathbf{T}$ admits quantifier elimination. □

1.5 Ultraproducts and ultrapowers

Definition 1.34. Let S be a non-empty set. A *filter* on S is a family $\mathcal{F}$ of subsets of S such that

(F1) $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;

(F2) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$;

(F3) if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq S$, then $Y \in \mathcal{F}$.

Example 1.35. Let S be a non-empty set and fix $\emptyset \neq X_0 \subseteq S$. The *principal filter generated by* X_0 is

$$\mathcal{F}(X_0) := \{X \subseteq S \mid X_0 \subseteq X\}.$$

Example 1.36. Let S be an infinite set. The *Frechet filter* on S is

$$\mathcal{F}_F := \{X \subseteq S \mid |S \setminus X| < \infty\}.$$

Observe that the Frechet filter is not a principal filter. Indeed,

$$\mathcal{E} := \{S \setminus \{a\} \mid a \in S\} \subseteq \mathcal{F}_f$$

and $\bigcap \mathcal{F}_F \subseteq \bigcap \mathcal{E} = \emptyset$. On the other hand, if $\mathcal{F}_F = \mathcal{F}(X_0)$ for some $\emptyset \neq X_0 \subseteq S$, then $\bigcap \mathcal{F}_F = X_0 \neq \emptyset$.

We begin with an easy observation.

Lemma 1.37. *Let S be a non-empty set.*

(*i*) *If $\mathscr{F}$ is a non-empty set of filters on S, then $\bigcap \mathscr{F}$ is a filter on S.*

(*ii*) *If $\mathscr{C}$ is a chain of filters on S, then $\bigcup \mathscr{C}$ is a filter on S.*

Definition 1.38. Let S be a non-empty set. A non-empty family $\mathcal{G}$ of subsets of S *has the finite intersection property* (FIP) if for all finite $\emptyset \neq \mathcal{H} \subseteq \mathcal{G}$ we have $\bigcap \mathcal{H} \neq \emptyset$.

Observe that any filter has the FIP. Moreover, if $\emptyset \neq \mathcal{G} \subseteq \mathcal{P}(S)$ has the FIP, then there exists a filter on S which contains $\mathcal{G}$. To see this one considers

$$\mathcal{F} := \left\{ X \subseteq S \mid \exists \mathcal{H} \subseteq \mathcal{G} \text{ finite and non-empty s.t. } \bigcap \mathcal{H} \subseteq X \right\}$$

and shows that it is a filter that contains $\mathcal{G}$.

Definition 1.39. A filter $\mathcal{U}$ on a non-empty set S is called an *ultrafilter* if for all $X \subseteq S$ we have $X \in \mathcal{U}$ or $S \setminus X \in \mathcal{U}$.

Lemma 1.40. *Let S be a non-empty set. A filter $\mathcal{U}$ on S is an ultrafilter if and only if it is maximal in $\mathcal{P}(\mathcal{P}(S))$ with respect to set-inclusion.*

Proof. Assume that $\mathcal{U}$ is an ultrafilter and suppose it is not maximal in $\mathcal{P}(\mathcal{P}(S))$. Then there exists a filter $\mathcal{F} \supsetneq \mathcal{U}$. Let $X \in \mathcal{F} \setminus \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $S \setminus X \in \mathcal{U}$. However, this implies $X, S \setminus X \in \mathcal{F}$ and then by (F2) we would obtain $\emptyset \in \mathcal{F}$ contradicting (F1).

Assume that $\mathcal{U}$ is maximal in $\mathcal{P}(\mathcal{P}(S))$ and suppose that there exists $X \subseteq S$ such that $X, S \setminus X \notin \mathcal{U}$. Let $\mathcal{G} := \mathcal{U} \cup \{X\}$. Then $\mathcal{G}$ has the FIP. Indeed, if $X \cap Y = \emptyset$ for some $Y \in \mathcal{U}$, then $Y \subseteq S \setminus X$ and thus $S \setminus X \in \mathcal{U}$, a contradiction. A filter which contains $\mathcal{G}$ would now contradict the maximality of $\mathcal{U}$. The proof is complete. □

By an application of Zorn's Lemma one can now derive the following result.

Proposition 1.41. *Let S be a non empty-set. Any filter on S is contained in an ultrafilter on S.*

Corollary 1.42. *Let S be an infinite set and let A be an infinite subset of S. There exists a non-principal ultrafilter $\mathcal{U}$ on S such that $A \in \mathcal{U}$.*

Proof. One considers $\mathcal{G} := \mathcal{F}_F \cup \{A\}$ and show that it has the FIP. One then obtains a filter $\mathcal{F}$ which contains $\mathcal{G}$ and extends it to the desired ultrafilter $\mathcal{U}$ using the previous proposition. Finally, one notes that $\mathcal{U}$ cannot be principal since it contains the Frechet filter $\mathcal{F}_F$. □

We now come to an important model theoretic application of ultrafilters. Let S be a non-empty set and $\mathcal{U}$ an ultrafilter on S. Assume that for all $s \in S$ we have an $\mathcal{L}$-structure $\mathfrak{A}^{(s)}$. We define an equivalence relation $\sim_{\mathcal{U}}$ on $\prod_{s\in S} A^{(s)}$ as follows:

$$(a^{(s)})_{s\in S} \sim_{\mathcal{U}} (b^{(s)})_{s\in S} \iff \{s \in S \mid a^{(s)} = b^{(s)}\} \in \mathcal{U}.$$

We consider the $\mathcal{L}$-structure

$$\mathfrak{A} := \prod_{s\in S} \mathfrak{A}^{(s)}/\mathcal{U}$$

consisting of:

- The universe of $\mathfrak{A}$ is

$$A := \prod_{s\in S} A^{(s)}/{\sim_{\mathcal{U}}} = \{[(a^{(s)})]_{\mathcal{U}} \mid a^{(s)} \in A^{(s)} \text{ for all } s \in S\}$$

- If $R \in \mathcal{R}$ is an n-ary relation symbol of $\mathcal{L}$, then

$$([(a_1^{(s)})]_{\mathcal{U}}, \ldots, [(a_n^{(s)})]_{\mathcal{U}}) \in R^{\mathfrak{A}} \Leftrightarrow \{s \in S \mid (a_1^{(s)}, \ldots, a_n^{(s)}) \in R^{\mathfrak{A}^{(s)}}\} \in \mathcal{U}.$$

- If $f \in \mathcal{F}$ is a function symbol of $\mathcal{L}$ we set

$$f^{\mathfrak{A}}([(a_1^{(s)})]_{\mathcal{U}}, \ldots, [(a_n^{(s)})]_{\mathcal{U}}) := [(f^{\mathfrak{A}^{(s)}}(a_1^{(s)}, \ldots, a_n^{(s)}))]_{\mathcal{U}}$$

- If $c \in \mathcal{C}$ is a constant symbol of $\mathcal{L}$, then

$$c^{\mathfrak{A}} = [(c^{\mathfrak{A}^{(s)}})]_{\mathcal{U}}.$$

Using the properties of ultrafilters one can show that the $\mathcal{L}$-structure $\mathfrak{A}$ is well-defined. It is called the *ultraproduct* of $\{\mathfrak{A}^{(s)} \mid s \in S\}$ with respect to the ultrafilter $\mathcal{U}$. The following is the foundamental theorem on ultraproducts. A proof can for instance be found in [43].

Theorem 1.43 (Łoś). *Let $\mathfrak{A}$ be the ultraproduct of $\{\mathfrak{A}^{(s)} \mid s \in S\}$ with respect to an ultrafilter $\mathcal{U}$ on S. For all $\mathcal{L}$-sentences φ*

$$\mathfrak{A} \models \varphi \quad \iff \quad \{s \in S \mid \mathfrak{A}^{(s)} \models \varphi\} \in \mathcal{U}$$

If $\mathfrak{A}^{(s)}$ is a fixed $\mathcal{L}$-structure for all $s \in S$, then we speak of *ultrapower*. The ultrapower of a structure $\mathfrak{A}$ with respect to an ultrafilter $\mathcal{U}$ on S will be denoted by $\mathfrak{A}^S/\mathcal{U}$.

Remark 1.44. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $B \subseteq A$. We consider the ultrapower with respect to an ultrafilter $\mathcal{U}$ over a set S of the $\mathcal{L}(B)$-structure $(\mathfrak{A}, B)$ and denote it by $(\mathfrak{A}, B)^*$. This is an $\mathcal{L}(B)$-structure. By definition of $(\mathfrak{A}, B)$, the parameters c_b $(b \in B)$ of $\mathcal{L}(B)$ are interpreted by b. It follows that in $(\mathfrak{A}, B)^*$ the parameters c_b are interpreted by $[(b)]_{\mathcal{U}}$.

This means that $(\mathfrak{A}, B)^*$ is the $\mathcal{L}(B)$-structure $(\mathfrak{A}^*, B)$ where c_b is interpreted as the class of the constant sequence $[(b)]_{\mathcal{U}} \in A^*$ for all $b \in B$.

By the theorem of Łoś, the ultrapower

$$\mathfrak{A}^* := \mathfrak{A}^S/\mathcal{U}$$

of an $\mathcal{L}$-structure $\mathfrak{A}$ with respect to an ultrafilter $\mathcal{U}$ on S can be viewed as an elementary extension of $\mathfrak{A}$.

Corollary 1.45. *With the notation introduced above, the map*

$$\iota : A \to A^*$$

defined by $\iota(a) = [(a)]_{\mathcal{U}}$ *for every* $a \in A$ *is an elementary embedding of* $\mathfrak{A}$ *in* $\mathfrak{A}^*$.

Proof. As we have seen in the previous remark, there is a canonical way to endow $\mathfrak{A}^*$ with an $\mathcal{L}(A)$-structure, where the interpretation of the constant symbol c_a for $a \in A$ is $[(a)]_{\mathcal{U}}$. Hence, by hypothesis ι is just the constant term relation between $(\mathfrak{A}, A)$ and $(\mathfrak{A}^*, A)$. In view of Part 4. of Lemma 1.28 we have to show that $(\mathfrak{A}, A) \models \varphi$ implies $(\mathfrak{A}^*, A) \models \varphi$ for every $\mathcal{L}(A)$-sentence φ. Since $(\mathfrak{A}^*, A)$ is just the ultrapower $(\mathfrak{A}, A)^S/\mathcal{U}$, we obtain from Łoś's Theorem that

$$(\mathfrak{A}, A) \models \varphi \Longrightarrow \{s \in S \mid (\mathfrak{A}, A) \models \varphi\} = S \in \mathcal{U} \Longrightarrow (\mathfrak{A}^*, A) \models \varphi. \qquad \square$$

We state without proving it the following remarkable theorem which we will use. For a proof see [5, Theorem 6.1.15].

Theorem 1.46. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* $\mathcal{L}$*-structures. Then* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are elementarily equivalent if and only if they have isomorphic ultrapowers.*

1.6 Examples

In this section we will present examples of first order languages, structures and theories that will be of interest for us.

First let us introduce the language $\mathcal{L}_{vf}$ of valued fields. It consists of the language of fields plus a unary relation symbol $\mathcal{O}$. We view a valued field (K, v) as an $\mathcal{L}_{vf}$-structure by interpreting $\mathcal{O}$ as the valuation ring of (K, v).

The theory of valued fields $\mathbf{T}_{vf}$ is the collection of all $\mathcal{L}_{vf}$-sentences given by the axioms of the theory of fields together with the following axioms which say that the interpretation of $\mathcal{O}$ is a valuation ring.

$$\begin{aligned}&\mathcal{O}(0) \wedge \mathcal{O}(1),\\&\forall x \forall y(\mathcal{O}(x) \wedge \mathcal{O}(y) \to \mathcal{O}(x-y) \wedge \mathcal{O}(xy)),\\&\forall x \forall y(xy = 1 \to \mathcal{O}(x) \vee \mathcal{O}(y)).\end{aligned}$$

Let us introduce a unary relation symbol $\mathcal{O}^\times$ which is definable in $\mathcal{L}_{vf}$ by

$$\mathcal{O}^\times(x) \quad \Longleftrightarrow \quad \mathcal{O}(x) \wedge \mathcal{O}(x^{-1}).$$

Let $\varphi(x_1, \ldots, x_n)$ be a formula in the language of fields. We construct an $\mathcal{L}_{vf}$-formula $\varphi_r(x_1, \ldots, x_n)$ such that for all $a_1, \ldots, a_n \in \mathcal{O}_v$

$$Kv \models \varphi(a_1 v, \ldots, a_n v) \quad \Longleftrightarrow \quad (K, v) \models \varphi_r(a_1, \ldots, a_n).$$

This is done by induction on the complexity of φ in the following way.

$$\begin{aligned}(t_1 = t_2)_r &:= \neg\mathcal{O}^\times(t_1 - t_2),\\(\neg\psi)_r &:= \neg\psi_r,\\(\psi \wedge \theta)_r &:= \psi_r \wedge \theta_r,\\(\forall x\psi)_r &:= \forall x(\mathcal{O}(x) \to \psi_r).\end{aligned}$$

Similarly, given a formula $\varphi(x_1, \ldots, x_n)$ in the language of ordered abelian groups we can construct an $\mathcal{L}_{vf}$-formula $\varphi_g(x_1, \ldots, x_n)$ such that for all $a_1, \ldots, a_n \in K^\times$

$$vK \models \varphi(va_1, \ldots, va_n) \quad \Longleftrightarrow \quad (K, v) \models \varphi_g(a_1, \ldots, a_n).$$

To do so one proceeds again by induction on the complexity of φ and sets

$$\begin{aligned}(t_1 = t_2)_g &:= \mathcal{O}^\times(t_1 t_2^{-1}),\\(t_1 < t_2)_g &:= \mathcal{O}(t_2 t_1^{-1}) \wedge \neg\mathcal{O}^\times(t_2 t_1^{-1}),\\(\neg\psi)_g &:= \neg\psi_g,\\(\psi \wedge \theta)_g &:= \psi_g \wedge \theta_g,\\(\forall x\psi)_g &:= \forall x(x \neq 0 \to \psi_g).\end{aligned}$$

Using these observations it is not difficult to express in $\mathcal{L}_{vf}$ the property of having residue characteristic 0. Let φ^p be the sentence in the language of fields stating that the characteristic is p, for some prime $p > 0$. Then the collection of all $\neg\varphi^p_r$ axiomatizes the theory of valued fields with residue characteristic 0 in $\mathcal{L}_{vf}$.

Another property of valued fields which can be axiomatized in $\mathcal{L}_{vf}$ is the property of being *henselian*. We will write $\mathcal{M}(x)$ instead of $\mathcal{O}(x) \wedge \neg\mathcal{O}^\times(x)$. A valued field is henselian if and only if it is a model of $\mathbf{T}_{vf}$ satisfying H_n for all $n \in \mathbb{N}$, where H_n is (the universal closure of)

$$\begin{aligned}\mathcal{O}(y) \wedge \bigwedge_{1\leq i\leq n} \mathcal{O}(x_i) \wedge \mathcal{M}(y^n + x_1y^{n-1} + \ldots + x_{n-1}y + x_n)\\ \wedge\mathcal{O}^\times(ny^{n-1} + (n-1)x_1y^{n-2} + \ldots + x_{n-1})\\ \longrightarrow \exists z : \mathcal{M}(y-z) \wedge z^n + x_1z^{n-1} + \ldots + x_{n-1}z + x_n = 0.\end{aligned}$$

The *language of valued hyperfields* $\mathcal{L}_{vh}$ is

$$\mathcal{L}_{vh} = \{r_+, -, 0, \cdot, ^{-1}, 1, \mathcal{O}\}$$

where $\{\cdot, ^{-1}, 1\}$ is the language of (multiplicative) groups, r_+ is a ternary relation symbol, $-$ is a unary relation symbol, 0 is a constant symbol and $\mathcal{O}$ is a unary relation symbol. The *language of hyperfields* $\mathcal{L}_{hf}$ is $\mathcal{L}_{vh}$ without the latter unary relation symbol. Note that these languages are extensions of the language of leading term structures (or, RV-structures) $\mathcal{L}_{RV} = \{r_+, 0, \cdot, ^{-1}, 1\}$, that have been studied by Flenner (see [17, 18]).

Informally, a hyperfield F is a field with a multivalued addition $+ : F \to \mathcal{P}^*(F)$ where $\mathcal{P}^*(F)$ is the family of all nonempty subsets of F. From the model theoretic point of view this can be encoded using the ternary relation symbol r_+ by setting

$$r_+(x, y, z) \iff z \in x + y.$$

We will formally introduce hyperfields and valued hyperfields in Chapter 2.

We will also consider some multi-sorted languages. These are a generalization of the concept of language we introduced above: the languages we considered so far are multi-sorted languages with only one sort. It is well-known that for any multi-sorted language there is a language (in our sense) with the same expressive power. For more details on multi-sorted languages we refer the reader to [38].

The *language $\mathcal{L}_{amc}$ of amc-structures* is a multi-sorted language with two sorts $\mathbf{A}$ and $\mathbf{M}$. For $\mathbf{A}$ we have the language of rings and for $\mathbf{M}$ we have the language of (multiplicative) groups. We further have a binary relation symbol $\Theta(x, y)$ of type $(\mathbf{A}, \mathbf{M})$, i.e., the first input x is from $\mathbf{A}$ and the second input y is from $\mathbf{M}$. We will formally introduce amc-structures in the second section of Chapter 4, see also [26].

The *Denef-Pas language* $\mathcal{L}_{DP}$ is a multi-sorted language for valued fields with an angular component map. It has three sorts: $\mathbf{VF}$, $\mathbf{VG}$ and $\mathbf{RF}$ with the language of fields for $\mathbf{VF}$ and $\mathbf{RF}$ and the language of ordered abelian groups $\{+, -, <, 0\}$ extended with a constant symbol ∞ for $\mathbf{VG}$. Moreover, one has a unary function symbol v of type $(\mathbf{VF}, \mathbf{VG})$ to be interpreted as a valuation from the universe of sort **VF** to the universe of sort **VG** and a unary function symbol α of type $(\mathbf{VF}, \mathbf{RF})$ to be interpreted as an angular component map (see Definition 4.12) from the universe of sort **VF** to the universe of sort **RF**.

Note that there are no relation symbols between different sorts in this language and no function symbols between **RF** and **VG**. A formula is said to be *of type* $(\mathbf{RF}, \mathbf{VG})$ if the only atomic formulae occuring in it are equalities in the **RF** sort or equalities in the **VG** sort or inequalities in the **VG** sort and it contains only quantifiers over the **RF** sort or over the **VG** sort. For example

$$\forall\gamma\exists\delta(\delta > \gamma) \wedge \neg(1 + 1 = 0)$$

is a formula of type $(\mathbf{RF}, \mathbf{VG})$, but

$$\forall\gamma\exists x(\gamma < vx) \wedge \forall x\neg(\alpha(x) + 1 = 0)$$

is not of type $(\mathbf{RF}, \mathbf{VG})$ since in it quantifiers over the **VF** sort occur. It follows that any $\mathcal{L}_{DP}$-formula φ of type $(\mathbf{RF}, \mathbf{VG})$ can be written as $\varphi_{\mathbf{RF}} \wedge \varphi_{\mathbf{VG}}$ where $\varphi_{\mathbf{RF}}$ is a formula in the language of fields and $\varphi_{\mathbf{VG}}$ is a formula in the language of ordered abelian groups extended with ∞.

Multi-sorted structures are defined in the same way as single-sorted ones. An $\mathcal{L}_{DP}$-structure $\mathfrak{K} = (K, Kv, vK, v, \alpha)$ is called *ac-valued field*. We will be interested in the $\mathcal{L}_{DP}$-theory $\mathbf{T}_{Pas}$ of henselian ac-valued fields with residue characteristic 0.

The following result was originally established by Pas in [41].

Theorem 1.47. *The theory* $\mathbf{T}_{Pas}$ *eliminates quantifiers over the sort* **VF**.

Chapter 2

Valued hyperfields

In this chapter we study the algebraic hyperstructures that are of interest for us in this work: valued hyperfields. Hyperfields are a generalization of the concept of fields where the addition is multivalued, i.e., adding two elements results in a non-empty subset of the hyperfield rather than a single element. These objects appeared for the first time in [24]. In that paper, Krasner also defines valued hyperfields. However, more recently Davvaz and Salasi in [13] gave a more general definition of valuation on hyperrings. We are going to use this more general notion.

In the first section of this chapter we will formally introduce hyperfields and study the notions of homorphism and substructure for them. We will also present Krasner's fundamental construction of the factor hyperfield of a field modulo a multiplicative subgroup of its multiplicative group. Since hyperfields are a particular kind of hyperrings, we will first define these and then also study the theory of hyperideals as developed by Jun in [28].

In the second section we will introduce valuations in the hyperfield setting. We will further note that a notion of valuation hyperring is natural to define and that this yields a correspondence between (equivalence classes of) valuations and valuation hyperrings as in the classical theory of valued fields.

This chapter contains joint work with K. Kuhlmann and H. Stojałowska (cf. [27] and [34]).

2.1 Hyperrings: homomorphisms, subhyperrings and hyperideals

Let H be a nonempty set and $\mathcal{P}^*(H)$ the family of nonempty subsets of H. A *hyperoperation* $+$ on H is a function which associates with every pair $(x, y) \in H \times H$ an element of $\mathcal{P}^*(H)$, denoted by $x + y$. A *hypergroupoid* is a nonempty set H

with a hyperoperation $+ : H \times H \to \mathcal{P}^*(H)$. For $x \in H$ and $A, B \subseteq H$ we set

$$A + B = \bigcup_{a \in A, b \in B} a + b, \tag{2.1}$$

$A + x = A + \{x\}$ and $x + A = \{x\} + A$.

In 1934 the notion of hypergroup was defined by F. Marty in [37] to be a hypergroupoid H with an associative hyperoperation $+$ (see Definition 2.1 below) such that $x + H = H + x = H$ for all $x \in H$. The following special class of hypergroups will be of interest for us.

Definition 2.1. A *canonical hypergroup* is a tuple $(H, +, 0)$, where $(H, +)$ is a hypergroupoid and 0 is an element of H such that the following axioms hold:

(H1) the hyperoperation $+$ is associative, i.e., $(x + y) + z = x + (y + z)$ for all $x, y, z \in H$,

(H2) $x + y = y + x$ for all $x, y \in H$,

(H3) for every $x \in H$ there exists a unique $x' \in H$ such that $0 \in x + x'$ (the element x' will be denoted by $-x$),

(H4) $z \in x + y$ implies $y \in z - x := z + (-x)$ for all $x, y, z \in H$.

Remark 2.2. Some authors in defining canonical hypergroups require explicitly that $x + 0 = \{x\}$ for all $x \in H$. However, we note that this axiom follows from (H3) and (H4). Indeed, suppose that $y \in x + 0$ for some $x, y \in H$. Then $0 \in y - x$ by (H4). Now $y = x$ follows from the uniqueness required in (H3). For this reason we call 0 the *neutral element for* $+$.

Remark 2.3. A canonical hypergroup is a hypergroup in the sense of Marty. Fix $a \in H$ and take $x \in H + a$. Then there exist $h \in H$ such that $x \in h + a \subseteq H$, showing that $H + a \subseteq H$. For the other inclusion, take $x \in H$, then

$$x \in x + 0 \subseteq x + (a - a) = (x - a) + a,$$

so there exists $h \in x - a \subseteq H$ such that $x \in h + a \subseteq H + a$.

Remark 2.4. Note that an abelian group $(G, +, 0)$ is not a priori a canonical hypergroup, because the operation on G is not a hyperoperation, as it takes values in G and not in $\mathcal{P}^*(G)$. However, it can be turned into a canonical hypergroup $(G, *, 0)$ by setting $x * y := \{x + y\}$ for all $x, y \in G$.

Definition 2.5. A *commutative hyperring* is a tuple $(R, +, \cdot, 0)$ which satisfies the following axioms:

(R1) $(R, +, 0)$ is a canonical hypergroup,

(R2) $(R, \cdot)$ is a commutative semigroup and 0 is an absorbing element, i.e., $x \cdot 0 = 0$ for all $x \in R$,

(R3) the operation $\cdot$ is distributive with respect to the hyperoperation $+$. That is, for all $x, y, z \in R$,
$$x \cdot (y + z) = x \cdot y + x \cdot z,$$

where we have set for $x \in R$ and $A \subseteq R$

$$xA := \{xa \mid a \in A\}.$$

If for all $x \in R$ we have that $xy = 0_R$ implies $x = 0_R$ or $y = 0_R$, then $(R, +, \cdot, 0_R)$ is called an *(integral) hyperdomain*. If the operation $\cdot$ has a neutral element $1_R \neq 0_R$, then we say that $(R, +, \cdot, 0_R, 1_R)$ is a *hyperring with unity*. If $(R, +, \cdot, 0_R, 1_R)$ is a hyperring with unity and $(R \setminus \{0_R\}, \cdot, 1_R)$ is an abelian group, then $(R, +, \cdot, 0_R, 1_R)$ is called a *hyperfield*.

Since we will not consider non-commutative hyperrings, in what follows we will refer to a commutative hyperring simply as a *hyperring*.

Remark 2.6. In the literature, the name "hyperring" can be found for structures $(R, +, \cdot)$ where both $+$ and $\cdot$ are hyperoperations. Some authors refer to the structures $(R, +, \cdot)$ with one hyperoperation $+$ and one operation $\cdot$ defined in Definition 2.5 as *Krasner's hyperrings*. There is also a particular hyperfield called *Krasner's hyperfield*, usually denoted by $\mathbb{K}$ (cf. [28, Example 2.5]). Furthermore, as Krasner introduced the hyperfields we will define in Definition 3.2, one may be tempted to call them "Krasner's hyperfields".

To avoid any confusion, since we will only consider structures as in Definition 2.5 above, we decided to reserve the names "hyperring" and "hyperfield" for them as indicated.

Remark 2.7. The double distributivity law, i.e.,

$$(a + b)(c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d,$$

does not hold in general in hyperrings and hyperfields. However, the following inclusion:
$$(a + b)(c + d) \subseteq ac + ad + bc + bd$$

holds. This was shown by Viro in [53, Section 4.4].

Examples of non-trivial hyperrings are described in [12, 4, 53]. Further, in [25] one can find the following construction which will be of interest for us.

Let A be a ring with unity and T a normal subgroup of its multiplicative semigroup (i.e., $xT = Tx$ for every $x \in A$). We consider the well-known equivalence relation

$$x \sim_T y \iff x = yt \text{ for some } t \in T$$

Denote by $[x]_T$ the equivalence class xT of $x \in A$ and by A_T the set of all equivalence classes.

Proposition 2.8. *The set A_T with the hyperoperation:*

$$[x]_T + [y]_T := \{[x + yt]_T \in A_T \mid t \in T\};$$

and the operation:

$$[x]_T \cdot [y]_T := [xy]_T,$$

is a hyperring with $[0]_T = \{0\}$ as neutral element for $+$. If A is a field, then A_T is a hyperfield with $[1]_T$ as neutral element for $\cdot$.

Proof. First we show that $(A_T, +, [0]_T)$ is a canonical hypergroup. The associative law follows from the same law in A. Indeed, after recalling (2.1), we observe that since T is a subgroup of the multiplicative semigroup of A, we have that

$$\begin{aligned} [x]_T + ([y]_T + [z]_T) &= \{[x + (y + zt)u]_T \in A_T \mid t, u \in T\} \\ &= \{[x + (yu + ztu)]_T \in A_T \mid t, u \in T\} \\ &= \{[x + (yt + zu)]_T \in A_T \mid t, u \in T\}. \end{aligned}$$

On the other hand,

$$([x]_T + [y]_T) + [z]_T = \{\{[(x + yt) + zu]_T \in A_T \mid t, u \in T\}\}.$$

Hence, $[x]_T+([y]_T+[z]_T) = ([x]_T+[y]_T)+[z]_T$ since the addition in A is associative.

The commutativity of $+$ follows from the commutativity of the addition in A.

The unique inverse of $[x]_T$ is $[-x]_T$. Indeed, since $1 \in T$, we obtain that

$$[x]_T + [-x]_T = \{[x - xt]_T \mid t \in T\} \ni [x - x]_T = [0]_T.$$

Moreover, if

$$[0]_T \in [x]_T + [y]_T = \{[x + yt]_T \mid t \in T\},$$

then there exist $t \in T$ such that $[0]_T = [x + yt]_T$. This means that there exists $u \in T$ such that $0 = (x + yt)u$. Multiplying by $u^{-1} \in T$, we obtain that $-x = yt$ and so $[-x]_T = [y]_T$. This shows (H3).

In order to show (H4) assume that $[z]_T \in [x]_T + [y]_T$. We wish to show that $[y]_T \in [z]_T - [x]_T$. We have

$$[z]_T \in \{[x + yt]_T \mid t \in T\},$$

so there exist $t \in T$ such that $[z]_T = [x + yt]_T$. This means that there exists $u \in T$ with $z = (x + yt)u = xu + ytu$. Hence, $ytu = z - xu$ which shows that $[ytu]_T = [y]_T \in [z]_T - [x]_T$ follows. We have shown that (R1) holds.

Since by definition $[x]_T[y]_T = [xy]_T$ and $(A, \cdot)$ is a commutative semigroup with $x0 = 0$ for all $x \in A$, (R2) follows.

It remains to show (R3). To this end, we observe that

$$\begin{aligned}([x]_T + [y]_T)[z]_T &= \{[x + yt]_T[z]_T \in A_T \mid t \in T\} \\ &= \{[xz + yzt]_T \in A_T \mid t \in T\} \\ &= [xz]_T + [yz]_T \\ &= [x]_T[z]_T + [y]_T[z]_T.\end{aligned}$$

For the last assertion, we first note that for all $[x]_T \in A_T$ we have that

$$[x]_T[1]_T = [x1]_T = [x]_T.$$

Thus, $[1]_T$ is the neutral element for the multiplication in A_T. If A is a field, then for all non-zero $[x]_T \in A_T$ we obtain

$$[x]_T[x^{-1}]_T = [xx^{-1}]_T = [1]_T.$$

Therefore, $[x]_T^{-1} = [x^{-1}]_T$. This completes the proof. □

The hyperrings constructed in the way described above are commonly called quotient hyperrings. However, to distinguish them from the hyperrings constructed in [28] (cf. Definition 2.22 below), which correspond to the classical construction of quotient rings modulo ideals, we will call them *factor hyperrings*.

For future reference, let us state the following simple observation.

Lemma 2.9. *In a factor hyperring A_T we have that $[z]_T \in [x]_T + [y]_T$ if and only if $z = xt + yu$ for some $t, u \in T$.*

Proof. If $[z]_T \in [x]_T + [y]_T$, then by definition of the hyperopoeration in A_T, there exists $t \in T$ such that $[z]_T = [x + yt]_T$. Therefore, there exists $u \in T$ such that $z = (x + yt)u = xu + ytu$. Since $tu \in T$, we have proved that $z = xt + yu$ for some $t, u \in T$.

Conversely, if $z = xt + yu$, then $zt^{-1} = x + yut^{-1}$, whence

$$[z]_T = [zt^{-1}]_T = [x + yut^{-1}]_T \in [x]_T + [y]_T.$$

Here we have used the fact that $t^{-1}, ut^{-1} \in T$. □

Example 2.10. Consider the field of real numbers $\mathbb{R}$ with its multiplicative subgroup $(\mathbb{R}^\times)^2$. We can identify the factor hyperfield $\mathbb{R}_{(\mathbb{R}^\times)^2}$ with the set $\{-1, 0, 1\}$. This hyperfield is called the *sign hyperfield*.

Definition 2.11. Let R and S be hyperrings. A map $\sigma : R \to S$ is a *homomorphism of hyperrings* if it satisfies

(HH1) $\sigma(0_R) = 0_S$;

(HH2) $\sigma(x \cdot_R y) = \sigma(x) \cdot_S \sigma(y)$ for all $x, y \in R$;

(HH3) $\sigma(x +_R y) \subseteq \sigma(x) +_S \sigma(y)$ for all $x, y \in R$.

An homomorphism of hyperrings σ is said to be *strict* if it satisfies the following property which is stronger than (HH3):

(HH3′) $\sigma(x +_R y) = (\sigma(x) +_S \sigma(y)) \cap \operatorname{Im} \sigma$ for all $x, y \in R$.

A strict bijective homomorphism of hyperrings is called an *isomorphism of hyperrings*. If there is an isomorphism of hyperrings $\sigma : R \to S$, then one says that R and S are *isomorphic* and writes $R \simeq S$.

Remark 2.12. Let $\mathcal{L}_{hr}$ be the language of hyperrings, i.e., $\mathcal{L}_{hf}$ without the unary function symbol $^{-1}$. Consider two hyperrings R, S as $\mathcal{L}_{hr}$-structures by interpreting $r_+(x, y, z)$ as $z \in x + y$. Then a morphism from R to S as in Definition 1.6 is a homomorphism of hyperrings and a strict morphism from R to S is a strict homomorphism of hyperrings. This justifies our terminology.

Let us mention that in the literature a different terminology is commonly used which we believe might be misleading. An homomorphism of hyperrings $\sigma : R \to S$ is said to be strict (or good) if it satisfies

(HH3″) $\sigma(x +_R y) = \sigma(x) +_S \sigma(y)$ for all $x, y \in R$.

We note that this notion does not coincide with the model theoretical notion of strict morphism, unless σ is surjective. If $\sigma : R \to S$ is an isomorphism of hyperrings, then (HH3″) holds.

Remark 2.13. It is clear that strict homomorphisms of hyperrings are homomorphisms of hyperrings.

Remark 2.14. Let $\sigma : R \to S$ be a homomorphism of hyperrings. It follows from the definition that $\sigma(-x) = -\sigma(x)$ for all $x \in R$. Indeed, for all $x \in R$ we have that $0_R \in x - x$ so that $0_S = \sigma(0_R) \in \sigma(x - x) \subseteq \sigma(x) + \sigma(-x)$. Hence, the uniqueness required in axiom (H3) implies that $\sigma(-x) = -\sigma(x)$.

The following observation will be useful later.

Lemma 2.15. *Let $(R, +_R, \cdot_R, 0_R)$ be a hyperring. Assume that a commutative semigroup $(S, \cdot_S, 0_S)$ is given with 0_S as an absorbing element. If $\sigma : R \to S$ is a bijection satisfying (HH1) and (HH2), then*

$$x +_S y := \sigma(\sigma^{-1}x +_R \sigma^{-1}y) \quad (x, y \in S)$$

defines a hyperoperation on S and $(S, +_S, \cdot_S, 0_S)$ is a hyperring. Moreover, σ is an isomorphism of hyperrings $R \simeq S$.

Proof. Clearly $+_S$ is a hyperoperation on S. We now wish to show (R1) for $(S, +_S, \cdot_S, 0_S)$. First, we have to prove that $+_S$ is associative. Take $x, y, z \in S$ and $a \in x +_S (y +_S z)$. There exists $b \in y +_S z$ such that $a \in x +_S b$. By definition of $+_S$ we have that $\sigma^{-1}b \in \sigma^{-1}y +_R \sigma^{-1}z$ and $\sigma^{-1}a \in \sigma^{-1}x +_R \sigma^{-1}b$. Therefore,

$$\sigma^{-1}a \in \sigma^{-1}x +_R (\sigma^{-1}y +_R \sigma^{-1}z) = (\sigma^{-1}x +_R \sigma^{-1}y) +_R \sigma^{-1}z.$$

This means that there exists $c' \in \sigma^{-1}x +_R \sigma^{-1}y$ such that $\sigma^{-1}a \in c' +_R \sigma^{-1}z$. Using the bijectivity of σ, we find $c \in S$ such that $c' = \sigma^{-1}c$. Hence,

$$c = \sigma c' \in \sigma(\sigma^{-1}x +_R \sigma^{-1}y) = x +_S y.$$

Moreover, it follows that $a = \sigma\sigma^{-1}a \in \sigma(\sigma^{-1}c +_R \sigma^{-1}z) = c +_S z$. This shows that $a \in (x +_S y) +_S z$, so $x +_S (y +_S z) \subseteq (x +_S y) +_S z$. The converse inclusion can be shown symmetrically.

Commutativity of $+_S$ is clear. For axiom (H3) we claim that for all $x \in S$ we have that $y := \sigma(-\sigma^{-1}x)$, where $-\sigma^{-1}x$ is the unique inverse of $\sigma^{-1}x$ in R, is the unique element of S such that $0_S \in x +_S y$. Indeed, we have $0_R \in \sigma^{-1}x +_R \sigma^{-1}y$, so $0_S = \sigma 0_R \in x +_S y$ and if $0_S \in x +_S y$, then

$$0_R = \sigma^{-1}0_S \in \sigma^{-1}(x +_S y) = \sigma^{-1}x +_R \sigma^{-1}y.$$

Hence, $\sigma^{-1}y = -\sigma^{-1}x$ in R and therefore $y = \sigma(-\sigma^{-1}x)$. Now it is straightforward to verify that the reversibility axiom (H4) in S follows from the same axiom in R.

By assumption on S the axiom (R2) of hyperrings holds. It then remains to show that the distributivity axiom (R3) holds. Take $x, y, z \in S$, we have to prove that

$$x(y +_S z) = xy +_S xz.$$

Since σ satisfies (HH2) and by definition of $+_S$, we obtain, using distributivity in R, that

$$\begin{aligned}\sigma^{-1}(x(y +_S z)) &= \sigma^{-1}x(\sigma^{-1}y +_R \sigma^{-1}z)\\ &= \sigma^{-1}(xy) +_R \sigma^{-1}(xz)\\ &= \sigma^{-1}(xy +_S xz).\end{aligned}$$

Now distributivity in S follows from the fact that σ is a bijection. We have shown that $(S, +_S, \cdot_S, 0_S)$ is a hyperring.

From the definition of $+_S$, it follows that σ^{-1} is a strict homomorphism of hyperrings $(S, +_S, \cdot_S, 0_S) \simeq (R, +_R, \cdot_R, 0_R)$. Therefore, σ is an isomorphism of hyperrings $R \simeq S$. □

Definition 2.16. Let $(R, +, \cdot, 0)$ be a hyperring. A subset S of R is a *subhyperring* of R if it is multiplicatively closed and with the *induced* hyperaddition

$$a +_S b := (a + b) \cap S \quad (a, b \in S)$$

is itself a hyperring.

A subset S of R is a *strict subhyperring* of R if $0 \in S$ and for all $a, b \in S$ one has that $a - b \subseteq S$ and $ab \in S$.

Remark 2.17. A strict subhyperring S of a hyperring $(R, +, \cdot, 0)$ is a subhyperring of R. However, there are examples of subhyperrings which are not strict subhyperrings (see [27, Example 3.6]).

Definition 2.18. Let R be a hyperring. A strict subhyperring I of R is a *hyperideal of* R if for every $z \in R$ and $x \in I$ we have that $zx \in I$.

Definition 2.19. Let $\sigma : R \to S$ be a homomorphism of hyperrings. The set

$$\ker \sigma := \{x \in R \mid \sigma(x) = 0_S\}$$

is called the *kernel* of σ.

Let R be a hyperring and I a hyperideal of R. Introduce the following relation on R:

$$x \sim_I y \iff x + I = y + I$$

where $x + I := \bigcup_{a \in I} x + a$ and the equality on the RHS is an equality of sets. Clearly, this is an equivalence relation. The next result is Lemma 3.3 in [28].

Lemma 2.20. *Let R be a hyperring and I a hyperideal of R. Then for all $x, y \in R$*

$$x \sim_I y \iff (x - y) \cap I \neq \emptyset$$

We will denote by $[x]_I$ the equivalence class of $x \in R$ under $\sim_I$. Let R be a hyperring and I a hyperideal of R. We let

$$R/I := \{[x]_I \mid x \in R\}$$

be the set of equivalence classes of $\sim_I$ on R. Further, for $x, y \in R$ we define

$$[x]_I + [y]_I := \{[z]_I \mid z \in x + y\}$$

and

$$[x]_I \cdot [y]_I := [xy]_I.$$

For the proof of the next result see [28, Proposition 3.5 and 3.6].

Proposition 2.21. *With the notation introduced above, we have that* $(R/I, +, \cdot, [0]_I)$ *is a hyperring. Moreover, the canonical projection map*

$$\begin{aligned} \pi_I : R &\to R/I \\ x &\mapsto [x]_I \end{aligned}$$

is a surjective strict homomorphism of hyperrings and $\ker \pi_I = I$.

Definition 2.22. We call the hyperring R/I constructed above the *quotient hyperring of R modulo I*.

In the case of hyperrings one recovers an analog of the classical correspondence between ideals and kernels of homomorphisms. More precisely, we have the following result.

Proposition 2.23. *Let R be a hyperring. The kernel of a homomoprhism of hyperrings is a hyperideal of R. Every hyperideal of R is the kernel of some strict homomoprhism of hyperrings.*

Proof. Let $\sigma : R \to S$ be a homomorphism of hyperrings. Take $x, y \in \ker \sigma$ and $r \in R$. By (HH3) we have that $\sigma(x - y) \subseteq \sigma(x) - \sigma(y) = \{0_S\}$ so $x - y \subseteq \ker \sigma$. Furthermore, $\sigma(rx) = \sigma(r)\sigma(x) = 0_S$. Hence, $rx \in \ker \sigma$ and $\ker \sigma$ is a hyperideal of R. To prove the second assertion, given a hyperideal I of R, we consider the canonical projection $\pi : R \to R/I$. By Proposition 2.21 we have that π is a strict homomoprhism of hyperrings and that $\ker \pi = I$. □

Lemma 2.24. *The intersection of hyperideals of a hyperring R is a hyperideal of R.*

Proof. Let $\mathcal{I}$ be a family of hyperideals of R. We have to show that $\bigcap \mathcal{I}$ is a hyperideal of R. It is clear that $0 \in \bigcap \mathcal{I}$ since $0 \in I$ for all $I \in \mathcal{I}$. Take $x, y \in \bigcap \mathcal{I}$ so that $x, y \in I$ for all $I \in \mathcal{I}$. Since every $I \in \mathcal{I}$ is a strict subhyperring of R we have $x - y \subseteq I$ and $xy \in I$ for all $I \in \mathcal{I}$. Therefore, $\bigcap \mathcal{I}$ is a strict subhyperring of R. Take $z \in R$ and $x \in \bigcap \mathcal{I}$ so that $x \in I$ for all $I \in \mathcal{I}$. Since every $I \in \mathcal{I}$ is a hyperideal of R we obtain that $zx \in I$ for all $I \in \mathcal{I}$, that is, $zx \in \bigcap \mathcal{I}$. This completes the proof. □

Let R be a hyperring. For a subset $X \subseteq R$, we denote by $\langle X \rangle$ the smallest hyperideal of R containing X. Note that $\langle X \rangle$ always exists by the lemma above. We call $\langle X \rangle$ the *hyperideal generated by X*. If $\{X_j : j \in J\}$ is a family of subsets of R indexed by an index set J, we denote by $\langle X_j \rangle_{j \in J}$ the hyperideal generated by $\bigcup_{j \in J} X_j$. The next proposition is Lemma 3.22 in [28]. A proof can be found in [30, Lemma 4.3.4].

Proposition 2.25. *Let R be a hyperring with unity.*

1. *For $a \in R$, the hyperideal generated by $\{a\}$ is*

$$aR := \{ax \mid x \in R\}.$$

2. *Let J be an index set. Suppose that $I_j := a_j R$ for some $a_j \in R$, for all $j \in J$. Then*

$$\langle I_j \rangle_{j \in J} = \left\{ x \in R \mid x \in \sum_{j \in X} r_j a_j,\ r_j \in R,\ X \subseteq J,\ |X| < \infty \right\}.$$

3. *Let J be an index set and $\{I_j : j \in J\}$ be a family of hyperideals of R. Then*

$$\langle I_j \rangle_{j \in J} = \left\{ x \in R \mid x \in \sum_{j \in X} r_j a_j,\ r_j \in R,\ a_j \in I_j,\ X \subseteq J,\ |X| < \infty \right\}.$$

Definition 2.26. Let R be a hyperring with unity. The set

$$R^\times := \{x \in R \mid \exists y \in R : xy = 1_R\}$$

is called the *set of units* of R.

The proof of the next lemma and its corollary are easy and similar to the proofs of their classical counterpart.

Lemma 2.27. *If a hyperideal I of a hyperring R contains a unit, then $I = R$.*

Corollary 2.28. *The only hyperideals of a hyperfield F are $\{0_F\}$ and F.*

Definition 2.29. Let I be a hyperideal of a hyperring R.

(i) I is called *prime* if for all $x, y \in R$ we have that $xy \in I$ implies $x \in I$ or $y \in I$.

(ii) I is called *maximal* if $I \subsetneq R$ and for all hyperideals J of R we have that $I \subsetneq J$ implies $J = R$.

In what follows we characterize prime and maximal hyperideals by means of quotient hyperrings as it is done in classical ring theory.

Proposition 2.30. *Let I be a hyperideal of a hyperring R with unity.*

(i) *I is prime if and only if R/I is a hyperdomain.*

(*ii*) *I is maximal if and only if R/I is a hyperfield.*

Proof. The proof of (*i*) is a copy of the corresponding proof in ring theory, since $x + I = I$ if and only if $(x - 0) \cap I \neq \emptyset$ which is equivalent to $x \in I$.

We come to the proof of (*ii*). Suppose that I is maximal. We have to show that R/I is a hyperfield, i.e., every nonzero $x + I \in R/I$ has a multiplicative inverse. Since $x + I$ is nonzero, we have that $x \notin I$. Consider the smallest hyperideal J of R which contains I and x, i.e., the intersection of all hyperideals of R which contain I and x. Since I is maximal we obtain that $J = R \ni 1$. By Part 3. of Proposition 2.25 we have

$$1 \in yx + i$$

for some $y \in R$ and $i \in I$. By axiom (H4) we obtain that $i \in (1-yx)$ and therefore $(1-yx)\cap I \neq \emptyset$. Thus, $1+I = yx+I$ by Lemma 2.20 and $y+I$ is a multiplicative inverse of $x + I$.

For the converse, we assume that R/I is a hyperfield and we take a hyperideal J of R such that $I \subsetneq J$. Take $x \in J \setminus I$. We have that $x + I$ is non-zero in R/I and hence there exists $y + I \in R/I$ such that

$$xy + I = (x + I)(y + I) = 1 + I.$$

Thus, $\emptyset \neq (1 - xy) \cap I \subseteq (1 - xy) \cap J$. Since $x \in J$ we have that $xy \in J$ and therefore $J = xy + J = 1 + J$. This shows that $1 \in J$ which implies that $J = R$ by Lemma 2.27. □

2.2 Valuation hyperrings and valuations

The definition of valuation hyperring in a hyperfield is natural.

Definition 2.31. Let F be a hyperfield. A subhyperring $\mathcal{O}$ of F is called a *valuation hyperring* if for all $x \in F \setminus \{0\}$ we have that either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

Observe that, by this definition, it follows that $1 \in \mathcal{O}$ for any valuation hyperring $\mathcal{O}$ in F.

Lemma 2.32. *A valuation hyperring $\mathcal{O}$ in a hyperfield F is a strict subhyperring of F.*

Proof. It suffices to show that $a - b \subseteq \mathcal{O}$ for all $a, b \in \mathcal{O}$. Take $a, b \in \mathcal{O}$ and $x \in a - b$. If $x \in \mathcal{O}$, then there is nothing to show (note that this case also includes $x = 0$). Otherwise, we have that $x^{-1} \in \mathcal{O}$ and thus $ax^{-1}, bx^{-1} \in \mathcal{O}$. Since $x \in a - b$ we obtain from (H4) that $a \in x + b$, so, using axiom (R3),

$$ax^{-1} \in (x + b)x^{-1} = 1 + bx^{-1}.$$

We have obtained that $ax^{-1} \in (1 + bx^{-1}) \cap \mathcal{O} = 1 +_{\mathcal{O}} bx^{-1}$. By (H4) and (R3) applied to the hyperring $(\mathcal{O}, +_{\mathcal{O}}, \cdot, 0)$, it follows that

$$xx^{-1} = 1 \in ax^{-1} +_{\mathcal{O}} (-bx^{-1}) = (a +_{\mathcal{O}} (-b))x^{-1}.$$

Therefore, $x \in a +_{\mathcal{O}} (-b) \subseteq \mathcal{O}$. This shows that $a - b \subseteq \mathcal{O}$. □

Lemma 2.33. *Let $\mathcal{O}$ be a valuation hyperring in a hyperfield F. Then $\mathcal{M} := \mathcal{O} \setminus \mathcal{O}^{\times}$ is the unique maximal hyperideal of $\mathcal{O}$.*

Proof. Take $a \in \mathcal{M}$ and $c \in \mathcal{O}$. If ca is invertible in $\mathcal{O}$, then there exists $x \in \mathcal{O}$ such that $x(ca) = 1$. Hence $(xc)a = 1$ and $a^{-1} = xc \in \mathcal{O}$ contradicting $a \in \mathcal{M}$. This proves that $ca \in \mathcal{M}$.

Take $a, b \in \mathcal{M}$. We may assume that $ab^{-1} \in \mathcal{O}$ (otherwise $ba^{-1} \in \mathcal{O}$ and we can interchange the roles of a and b). Since $\mathcal{O}$ is a strict hyperring we obtain that $1 - ab^{-1} \subseteq \mathcal{O}$ and therefore, using what we just proved above,

$$b - a = b(1 - ab^{-1}) \subseteq \mathcal{M}.$$

We have shown that $\mathcal{M}$ is a hyperideal of $\mathcal{O}$.

Since, by the definition of $\mathcal{M}$, we have that $\mathcal{O} \setminus \mathcal{M} = \mathcal{O}^{\times}$, by Lemma 2.27, every proper hyperideal of $\mathcal{O}$ must be contained in $\mathcal{M}$, showing that $\mathcal{M}$ is the unique maximal hyperideal of $\mathcal{O}$. □

Definition 2.34. Let F be a hyperfield, $\mathcal{O}$ a valuation hyperring in F and $\mathcal{M}$ its unique maximal hyperideal. By Proposition 2.30, the quotient hyperring $\mathcal{O}/\mathcal{M}$ is a hyperfield called the *residue hyperfield*.

The next definition was given for hyperrings by Davvaz and Salasi in [13, Definition 4.2].

Definition 2.35. Take a hyperfield F and an ordered abelian group Γ (written additively). A surjective map $v : F \to \Gamma \cup \{\infty\}$ is called a *valuation on F* if it has the following properties:

(V1) $va = \infty \iff a = 0$;

(V2) $v(ab) = va + vb$;

(V3) $c \in a + b \implies vc \geq \min\{va, vb\}$.

If v is a valuation on a hyperfield F we call (F, v) a *valued hyperfield*.

Lemma 2.36. *Let $v : F \to \Gamma \cup \{\infty\}$ be a valuation on a hyperfield F. Then:*

1. $v(1) = v(-1) = 0$,

2. *$v(-x) = vx$ for all $x \in F$,*

3. *$vx^{-1} = -vx$ for all $x \in F$,*

4. *if $vx \neq vy$, then for every $z \in x + y$, $vz = \min\{vx, vy\}$.*

Proof. For all $x \in F$ we have $vx = v(x \cdot 1) = vx + v(1)$, therefore $v(1) = 0$. Further, $0 = v(1) = v((-1)(-1)) = v(-1) + v(-1)$. Thus, $v(-1) = 0$.

For $x \in F$ we have $-x = (-1)x$, whence $v(-x) = v(-1) + vx = vx$.

If $x \in F$, then $0 = v(1) = v(xx^{-1}) = vx + vx^{-1}$. Therefore, $vx^{-1} = -vx$.

Finally, assume that $x, y \in F$ are such that $vx \neq vy$. Without loss of generality $vx < vy$ and for $z \in x + y$ we obtain $x \in z - y$. Suppose that $vz > \min\{vx, vy\} = vx$. By (V3) and Part 2. we have

$$vx \geq \min\{vz, v(-y)\} = \min\{vz, vy\} > vx.$$

This contradiction completes the proof. □

The next results shows that constructions analogous to classical ones can be carried out in the hyperfield setting.

Proposition 2.37. *Let $v : F \to \Gamma \cup \{\infty\}$ be a valuation on a hyperfield F. Then*

$$\mathcal{O}_v := \{x \in F \mid vx \geq 0\}$$

is a valuation hyperring in F and

$$\mathcal{M}_v := \{x \in F \mid vx > 0\}$$

is its unique maximal hyperideal.

Proof. We first prove that $\mathcal{O}_v$ is a (strict) subhyperring of F. Take $a, b \in \mathcal{O}_v$. By (V3), for all $c \in a - b$ we have $vc \geq \min\{va, v(-b)\} = \min\{va, vb\} \geq 0$, so $a - b \subseteq \mathcal{O}_v$. Further, we have $ab \in \mathcal{O}_v$ by (V2). By Part 3. of Lemma 2.36 we conclude that if $x \notin \mathcal{O}_v$, then $x^{-1} \in \mathcal{O}_v$ so $\mathcal{O}_v$ is a valuation hyperring in F.

Next we show that $\mathcal{M}_v$ is the unique maximal hyperideal of $\mathcal{O}_v$. Observe that, by virtue of Lemma 2.36,

$$\mathcal{O}_v^\times = \{x \in \mathcal{O}_v \mid vx = 0\}.$$

Hence, $\mathcal{M}_v = \mathcal{O}_v \setminus \mathcal{O}_v^\times$ and then $\mathcal{M}_v$ is the unique maximal hyperideal of $\mathcal{O}_v$ by Lemma 2.33. □

If a valuation v is given on a hyperfield F, we denote by vF the value group $v(F^\times)$ and by Fv the residue hyperfield $\mathcal{O}_v/\mathcal{M}_v$.

Remark 2.38. In the literature other attempts have been made to define the residue of a valued hyperfield. We postpone the discussion of this to Remark 3.27 below.

Remark 2.39. There are valued hyperfields with a residue hyperfield which is not a field. For an example see Example 3.28 below.

As in classical valuation theory, any valuation hyperring in a hyperfield F induces a canonical valuation on F.

Proposition 2.40. *Let F be a hyperfield and $\mathcal{O}$ a valuation hyperring in F. Consider the multiplicative group $\Gamma := F^\times/\mathcal{O}^\times$ and define a relation $\leq$ on Γ as follows:*

$$a\mathcal{O}^\times \leq b\mathcal{O}^\times \iff ba^{-1} \in \mathcal{O}.$$

Then $(\Gamma, \cdot, \leq)$ is an ordered abelian group and the canonical projection

$$\pi : F \to \Gamma \cup \{\infty\},$$

extended so that $\pi(0_F) = \infty$, is a valuation on F. Furthermore, $\mathcal{O}_\pi = \mathcal{O}$.

Proof. First we show that $\leq$ is an ordering for $(\Gamma, \cdot)$. Since $aa^{-1} = 1_F \in \mathcal{O}$, reflexivity is clear. If $ab^{-1}, ba^{-1} \in \mathcal{O}$, then $ab^{-1} \in \mathcal{O}^\times$ so $a\mathcal{O}^\times = b\mathcal{O}^\times$. Hence $\leq$ is antisymmetric. If $ab^{-1}, bc^{-1} \in \mathcal{O}$, then $ac^{-1} = ab^{-1}bc^{-1} \in \mathcal{O}$, showing that $\leq$ is transitive. Take now $a, b \in F^\times$ such that $a\mathcal{O}^\times \leq b\mathcal{O}^\times$ and $c \in F^\times$. We have that $bc(ac)^{-1} = bcc^{-1}a^{-1} = ba^{-1} \in \mathcal{O}$, whence $ac\mathcal{O}^\times \leq bc\mathcal{O}^\times$. This shows that $\leq$ is compatible with the operation of Γ. Finally, $\leq$ is a total order since $\mathcal{O}$ is a valuation hyperring, so that $ab^{-1} \in \mathcal{O}$ or $ba^{-1} \in \mathcal{O}$ for all $a, b \in F^\times$.

We now show that π is a valuation on F. Clearly, π is a surjective map, onto the ordered abelian group Γ with ∞ and (V1) holds. Since π is a homomorphism of groups we obtain (V2). It remains to show that (V3) holds for π. Take $x, y \in F$. If one of them is 0_F, then (V3) is straightforward. We may then assume that $x, y \in F^\times$ and that $x\mathcal{O}^\times \leq y\mathcal{O}^\times$. Take $z \in x + y$. We wish to show that $zx^{-1} \in \mathcal{O}$. By assumption we have that $yx^{-1} \in \mathcal{O}$, thus

$$zx^{-1} \in (x + y)x^{-1} = 1 + yx^{-1} \subseteq \mathcal{O},$$

where we used Lemma 2.32.

Finally, we observe that, by definition

$$\mathcal{O}_\pi = \{x \in F \mid \pi x \geq 1\} = \{x \in F \mid x\mathcal{O}^\times \geq 1_F\mathcal{O}^\times\} = \{x \in F \mid x \in \mathcal{O}\} = \mathcal{O}. \quad \square$$

Definition 2.41. For $i = 1, 2$ let $v_i : F \to \Gamma_i \cup \{\infty\}$ be valuations on a hyperfield F. We say that v_1 and v_2 are *equivalent* if there exists an isomorphism of ordered abelian groups $\sigma : \Gamma_1 \to \Gamma_2$ such that $v_2 = \sigma \circ v_1$.

Lemma 2.42. *Let $v : F \to \Gamma \cup \{\infty\}$ be a valuation on a hyperfield F. Then $\Gamma \simeq F^\times/\mathcal{O}_v^\times$ as ordered abelian groups.*

Proof. We consider $F^\times/\mathcal{O}_v^\times$ as an ordered abelian group with the ordering defined in Proposition 2.40. Using the surjectivity of v, we define a map

$$\sigma : \Gamma \to F^\times/\mathcal{O}_v^\times$$

by $\sigma(va) = a\mathcal{O}_v^\times$ for all $a \in F^\times$. This is well-defined since if $va = vb$, then $va - vb = v(ab^{-1}) = 0$ so that $ab^{-1} \in \mathcal{O}_v^\times$ and then $a\mathcal{O}_v^\times = b\mathcal{O}_v^\times$. Using property (V2) of valuations, we obtain that σ is a homomorphism of groups. Further, if $va \leq vb$, then $ba^{-1} \in \mathcal{O}_v$ which means that $\sigma(va) \leq \sigma(vb)$. Thus, σ is order-preserving. It is clear that σ is surjective. It therefore remains to show that σ is injective. To this end, assume that $a\mathcal{O}_v^\times = b\mathcal{O}_v^\times$ for some $a, b \in F^\times$. Then there exists $c \in \mathcal{O}_v^\times$ such that $a = bc$. Since $vc = 0$, by (V2) we obtain that $va = vb$. This completes the proof. □

Remark 2.43. Observe that by construction of σ in the above proof, we have that $\sigma \circ v = \pi$ where π is the canonical projection $F^\times \to F^\times/\mathcal{O}_v^\times$.

Corollary 2.44. *For $i = 1, 2$ let $v_i : F \to \Gamma_i \cup \{\infty\}$ be valuations on a hyperfield F. Then v_1 and v_2 are equivalent if and only if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.*

Proof. By the previous lemma we obtain for $i = 1, 2$ that $\Gamma_i \simeq F^\times/\mathcal{O}_{v_i}^\times$ as ordered abelian groups with isomorphisms σ_i such that $\sigma_i \circ v_i = \pi_i$ where $\pi_i : F^\times \to F^\times/\mathcal{O}_{v_i}^\times$ is the canonical projection for $i = 1, 2$. Thus, if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$, then $\pi_1 = \pi_2$ and $\sigma := \sigma_2^{-1} \circ \sigma_1$ is an isomorphism of ordered abelian groups $\Gamma_1 \to \Gamma_2$. Further we have that

$$\sigma \circ v_1 = \sigma_2^{-1} \circ (\sigma_1 \circ v_1) = \sigma_2^{-1} \circ \pi_2 = v_2.$$

Hence, v_1 and v_2 are equivalent.

On the other hand, if v_1 and v_2 are equivalent, then we obtain that $F^\times/\mathcal{O}_{v_1}^\times \simeq F^\times/\mathcal{O}_{v_2}^\times$ as ordered abelian groups. In particular, for $a \in F^\times$ we have that $1\mathcal{O}_{v_1}^\times \leq a\mathcal{O}_{v_1}^\times$ if and only if $1\mathcal{O}_{v_2}^\times \leq a\mathcal{O}_{v_2}^\times$. Using the definition of the ordering in $F^\times/\mathcal{O}_{v_i}^\times$ we see that this means that $a \in \mathcal{O}_{v_1}$ if and only if $a \in \mathcal{O}_{v_2}$. Since $0 \in \mathcal{O}_{v_i}$ for $i = 1, 2$, we conclude that $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$ as claimed. □

In what follows we will always consider valuations on hyperfields up to equivalence: for a valuation v we will refer to the equivalence class of the valuation under the equivalence relation defined above.

Chapter 3

Valued hyperfields associated to valued fields

In this chapter, we investigate more on some ideas of Krasner.

In the first section, we note that some factor hyperfields of a valued field naturally inherits a valuation becoming valued hyperfields in our sense. We then define some particular valued hyperfields which are associated to any valued field. These include the valued hyperfields that Krasner had in mind and have been studied also by Lee in [32] and by Tolliver in [48]. We study their basic properties. The second section is intended to give some motivation for the study of these objects. In the third section we investigate the residue hyperfield of these particular valued hyperfields. The fourth section is mainly devoted to the relations between the definition of valued hyperfields given by Krasner and the one introduced by Davvaz and Salasi which we use. In the last section of this chapter, we interpret in a modern category theoretical setting, an idea of Krasner on the inverse limit of the valued hyperfields associated to a complete valued field.

This chapter contains original work combined with the work of Krasner, Lee and Tolliver. Precise reference to their work is given throughout the presentation.

3.1 Factor hyperfields of valued fields

We begin with the following easy but fundamental observation.

Proposition 3.1. *Let (K, v) be a valued field and take a subgroup T of $K^\times$. If $T \subseteq \mathcal{O}_v^\times$, then*

$$\begin{aligned} v_T : K_T &\to vK \cup \{\infty\} \\ [x]_T &\mapsto vx \end{aligned}$$

is a valuation on K_T.

Proof. First we show that v_T is well-defined. Take $x, y \in K$ such that $[x]_T = [y]_T$. Then there exists $t \in T$ such that $x = yt$. Since $T \subseteq \mathcal{O}_v^\times$, we obtain that $vt = 0$. Thus,

$$v_T[x]_T = vx = v(yt) = vy + vt = vy = v_T[y]_T.$$

It is clear that v_T is surjective and that (V1) holds. Since by definition $[x]_T[y]_T = [xy]_T$ for all $[x]_T, [y]_T \in F$, it follows that

$$v_T([x]_T[y]_T) = v(xy) = vx + vy = v_T[x]_T + v_T[y]_T,$$

which is (V2).

In order to show (V3) we take $[z]_T \in [x]_T + [y]_T$. By Lemma 2.9, $z = xt + yu$ for some $t, u \in T$. Thus, since by assumption $vt = vu = 0$, we obtain that

$$v_T[z]_T = v(xt + yu) \geq \min\{v(xt), v(yu)\} = \min\{vx, vy\} = \min\{v_T[x]_T, v_T[y]_T\}.$$

which is (V3). □

Consider a valued field (K, v) and let $\gamma \in vK$ be a non-negative element of the value group. We set

$$\mathcal{M}_v^\gamma = \mathcal{M}^\gamma := \{x \in K \mid vx > \gamma\}.$$

Then $\mathcal{M}^\gamma$ is an ideal of $\mathcal{O}_v$ and $\mathcal{M}^0$ corresponds to the unique maximal ideal $\mathcal{M}_v$ of $\mathcal{O}_v$. The set of *1-units of level* γ will be for us the coset $1 + \mathcal{M}^\gamma$. Note that $1 + \mathcal{M}^\gamma$ is a subgroup of $K^\times$, thus we may consider the factor hyperfield

$$\mathcal{H}_\gamma(K) := K_{1+\mathcal{M}^\gamma}.$$

We denote by $[x]_\gamma$ the equivalence class of $x \in K$ in $\mathcal{H}_\gamma(K)$. Our notation here is in accordance with the notation adopted by Lee in [32]. Since $1 + \mathcal{M}^\gamma \subseteq \mathcal{O}_v^\times$, by Proposition 3.1 we obtain a valuation

$$\begin{aligned} v_\gamma : \mathcal{H}_\gamma(K) &\to vK \cup \{\infty\} \\ [x]_\gamma &\mapsto vx \end{aligned}$$

induced by the valuation v on K.

Definition 3.2 (Definition 2.7 in [32])**.** The valued hyperfield $(\mathcal{H}_\gamma(K), v_\gamma)$ is called the *γ-valued hyperfield* of K.

For the sake of completeness we state and prove the following very useful result which is Lemma 3.1 in [32].

Lemma 3.3 (Lee's Lemma). *Let (K, v) be a valued field. Take $a, b \in K$ and $a_0, \ldots, a_k \in K$ ($k \in \mathbb{N}$). Fix $\gamma \in vK$ such that $\gamma \geq 0$. The following assertions hold:*

1) If $a \neq 0$, then $[a]_\gamma = \{x \in K \mid v(x-a) > \gamma + va\}$.

2) If a and b are not both 0, then

$$\bigcup([a]_\gamma + [b]_\gamma) = \{x \in K \mid v(x-(a+b)) > \gamma + \min\{va, vb\}\}.$$

3) If a and b are not both 0, then

$$0 \in \bigcup([a]_\gamma + [b]_\gamma) \Longleftrightarrow \bigcup([a]_\gamma + [b]_\gamma) = \mathcal{M}^{\gamma+\min\{va,vb\}}.$$

4) $a_0 + \ldots + a_k \in \bigcup([a_0]_\gamma + \ldots + [a_k]_\gamma)$.

5) Suppose that $b \in \bigcup([a_0]_\gamma + \ldots + [a_k]_\gamma)$ and that $a_0, \ldots, a_k \in \mathcal{O}_v$ are not all 0. Then $b = (a_0 + \ldots + a_k) + d$ for some $d \in \mathcal{M}^\gamma$.

Proof. To prove 1) observe that by definition of $[a]_\gamma$ we have that $x \in [a]_\gamma$ if and only if $x = a + ad$ for some $d \in \mathcal{M}^\gamma$. This means that

$$v(x-a) = v(ad) = va + vd > va + \gamma.$$

We now come to the proof of 2). Take $x \in \bigcup([a]_\gamma + [b]_\gamma)$. This means that $[x]_\gamma \in [a]_\gamma + [b]_\gamma$, therefore by Lemma 2.9, $x = at + bu$ for some $t, u \in 1 + \mathcal{M}^\gamma$. Thus, there are $c, d \in \mathcal{M}^\gamma$ such that $x = a(1+c) + b(1+d) = a + b + ac + bd$. We obtain that

$$v(x-(a+b)) = v(ac+bd) \geq \min\{v(ac), v(bd)\} > \gamma + \min\{va, vb\}.$$

This shows that $\bigcup([a]_\gamma + [b]_\gamma) \subseteq \{x \in K \mid v(x-(a+b)) > \gamma + \min\{va, vb\}\}$. For the other inclusion, take $x \in K$ such that $v(x-(a+b)) > \gamma + \min\{va, vb\}$. Without loss of generality assume that $va \geq vb$. There is $c \in \mathcal{M}^\gamma$ such that

$$x = (a+b) + bc = a + b(1+c).$$

Namely, $c := b^{-1}(x-(a+b))$. We conclude that $[x]_\gamma \in [a]_\gamma + [b]_\gamma$. Therefore, $x \in \bigcup([a]_\gamma + [b]_\gamma)$. This shows the converse inclusion.

To show 3) apply 2) to obtain that $\bigcup([a]_\gamma + [b]_\gamma) = a + b + \mathcal{M}^{\gamma+\min\{va,vb\}}$ as sets. Thus,

$$\begin{aligned} 0 \in \bigcup([a]_\gamma + [b]_\gamma) &\Longleftrightarrow v(a+b) > \gamma + \min\{va, vb\} \\ &\Longleftrightarrow a + b \in \mathcal{M}^{\gamma+\min\{va,vb\}} \\ &\Longleftrightarrow \bigcup([a]_\gamma + [b]_\gamma) = \mathcal{M}^{\gamma+\min\{va,vb\}}. \end{aligned}$$

To show 4) we proceed by induction on k. If $k = 1$, then by 2) we obtain that $a_0 + a_1 \in \bigcup [a_0]_\gamma + [a_1]_\gamma$ since $v(a_0 + a_1 - (a_0 + a_1)) = \infty > \gamma + \min\{va_0, va_1\}$. Assume now that for $k > 1$ we have that

$$a_0 + \ldots + a_k \in \bigcup ([a_0]_\gamma + \ldots + [a_k]_\gamma).$$

We have to show that

$$a_0 + \ldots + a_{k+1} \in \bigcup ([a_0]_\gamma + \ldots + [a_{k+1}]_\gamma).$$

By definition (cf. (2.1)) we have that

$$[a_0]_\gamma + \ldots + [a_{k+1}]_\gamma = \bigcup_{[x]_\gamma \in [a_0]_\gamma + \ldots + [a_k]_\gamma} [x]_\gamma + [a_{k+1}]_\gamma \ .$$

Hence, by the induction hypothesis we obtain that

$$[a_0 + \ldots + a_k]_\gamma + [a_{k+1}]_\gamma \subseteq [a_0]_\gamma + \ldots + [a_{k+1}]_\gamma \ .$$

Now $[a_0 + \ldots + a_{k+1}]_\gamma \in [a_0 + \ldots + a_k]_\gamma + [a_{k+1}]_\gamma$ follows from 2).

In order to show 5) we proceed again by induction on k. If $k = 1$, then by 2) we have that

$$b \in \bigcup ([a_0]_\gamma + [a_1]_\gamma)$$

if and only if $v(b - (a_0 + a_1)) > \gamma + \min\{va_0, va_1\} \geq \gamma$. Therefore, $b = a_0 + a_1 + d$ for some $d \in \mathcal{M}^\gamma$, namely, $d := b - (a_0 + a_1)$. Assume now that assertion 5) is true for $k > 1$. Take

$$b \in \bigcup ([a_0]_\gamma + \ldots + [a_{k+1}]_\gamma).$$

By definition (cf. (2.1)), there exists $[x]_\gamma \in [a_0]_\gamma + \ldots + [a_k]_\gamma$ such that $[b]_\gamma \in [x]_\gamma + [a_{k+1}]_\gamma$. Now by 2) this means that

$$v(b - x - a_{k+1}) > \gamma + \min\{vx, va_{k+1}\} \tag{3.1}$$

and by the induction hypothesis we find $c \in \mathcal{M}^\gamma$ such that $x = a_0 + \ldots + a_k + c$. Note that the latter implies that $vx \geq 0$, whence from (3.1) we obtain that

$$v(b - (a_0 + \ldots + a_{k+1}) - c) > \gamma$$

Since $c \in \mathcal{M}^\gamma$, this implies that

$$v(b - (a_0 + \ldots + a_{k+1})) > \gamma$$

so that $b = a_0 + \ldots + a_{k+1} + d$ for some $d \in \mathcal{M}^\gamma$, as required. □

Let (K, v) be a valued field and $\gamma \in vK$ be a non-negative element of the value group. For technical reasons let us define the language $\mathcal{L}_{vf}^{\gamma}$ as $\mathcal{L}_{vf}$ extended with a constant symbol c to be interpreted as an element of value γ.

Corollary 3.4. *For every $\mathcal{L}_{vh}$-formula $\varphi = \varphi(x_1, \ldots, x_n)$ there is an $\mathcal{L}_{vf}^{\gamma}$-formula $\varphi_h = \varphi_h(x_1, \ldots, x_n)$ such that for all valued fields (K, v) and all $a_1, \ldots, a_n \in K$ we have that*

$$(\mathcal{H}_\gamma(K), v_\gamma) \models \varphi([a_1]_\gamma, \ldots, [a_n]_\gamma) \iff (K, v) \models \varphi_h(a_1, \ldots, a_n)$$

Proof. We proceed by induction on the complexity of φ.

If φ is an equality $[a]_\gamma = [b]_\gamma$, then in the case $a, b \neq 0$, φ_h is $\mathcal{M}((1 - ab^{-1})c^{-1})$. This works since $\mathcal{M}((1 - ab^{-1})c^{-1})$ is equivalent to $v(1 - ab^{-1}) - vc > 0$ and hence equivalent to $ab^{-1} \in 1 + \mathcal{M}^\gamma$ which holds if and only if $[a]_\gamma = [b]_\gamma$. If $a = 0$, then φ_h is $b = 0$ and symmetrically, if $b = 0$, then φ_h is $a = 0$.

If φ says that $[a]_\gamma$ is in the valuation hyperring of $\mathcal{H}_\gamma(K)$, then φ_h says that a is in the valuation ring of (K, v). This works because by definition $v_\gamma[x]_\gamma = vx$ for all $x \in K$.

If φ is $r_+([a]_\gamma, [b]_\gamma, [d]_\gamma)$, then in the case $a, b \neq 0$ we can set φ_h to be

$$(\mathcal{O}(ab^{-1}) \rightarrow \mathcal{M}((d - (a + b))c^{-1}b^{-1})) \wedge (\mathcal{O}(ba^{-1}) \rightarrow \mathcal{M}((d - (a + b))c^{-1}a^{-1})).$$

That this works follows by Part 2) of Lemma 3.3. If $a = 0$ and $b \neq 0$, then $[a]_\gamma + [b]_\gamma = \{[b]_\gamma\}$ is a singleton, so φ_h is $([d]_\gamma = [b]_\gamma)_h$. The case $b = 0$ and $a \neq 0$ is analogous. If $a = b = 0$, then φ_h is $d = 0$.

If φ is $\psi \wedge \theta$, then φ_h is $\psi_h \wedge \theta_h$, if φ is $\neg\psi$, then φ_h is $\neg\psi_h$. Finally, if φ_h is $\forall x \psi$, then φ_h is $\forall x \psi_h$. □

Remark 3.5. There are some situations where the constant extension $\mathcal{L}_{vf}^{\gamma}$ of $\mathcal{L}_{vf}$ is not needed. For instance, assume that we are working with the 0-valued hyperfield. Then since 1 is always a constant symbol in the language $\mathcal{L}_{vf}$ and $v1 = 0$, one can replace the c by 1 in the proof above thereby obtaining an $\mathcal{L}_{vf}$-formula φ_h.

Another similar situation happens when K is of characteristic 0 and we are working with the vk-valued hyperfield for some $k \in \mathbb{N} \subseteq K$. Then one can replace c by the element

$$\underbrace{1 + \ldots + 1}_{k \text{ times}}$$

in the proof above thereby obtaining an $\mathcal{L}_{vf}$-formula φ_h.

3.2 Why use hyperoperations?

In the beginning of this section we restrict our attention to the 0-valued hyperfield of a valued field (K, v). We make the following observation.

Lemma 3.6. *Let (K, v) be a valued field. Take $x, y \in K$ such that $[0]_0 \notin [x]_0 + [y]_0$. Then $[x]_0 + [y]_0 = \{[x + y]_0\}$.*

Proof. If $x = 0$ or $y = 0$, then the result follows trivially. We may then assume that $x, y \in K^\times$. By Part 2) of Lemma 3.3 the assumption $[0]_0 \notin [x]_0 + [y]_0$ implies that $v(x + y) \leq \min\{vx, vy\}$. However, we know that $v(x + y) \geq \min\{vx, vy\}$ always. Therefore, $v(x + y) = \min\{vx, vy\}$ must hold. By Part 2) of Lemma 3.3, $[z]_0 \in [x]_0 + [y]_0$ if and only if $v(z - (x + y)) > \min\{vx, vy\} = v(x + y)$. By Part 1) of Lemma 3.3, this means that $z \in [x + y]_0$ so that $[z]_0 = [x + y]_0$ as we wished to show. □

Remark 3.7. In the literature (see for instance [4]) hyperfields which satisfy the property that $a + b$ is a singleton unless $0 \in a + b$ are called *stringent.*

One may be temped to define the following operation on $\mathcal{H}_0(K)$:

$$[x]_0 * [y]_0 := \begin{cases} [x + y]_0 & \text{if } [0]_0 \notin [x]_0 + [y]_0, \\ [0]_0 & \text{otherwise.} \end{cases}$$

to avoid the use of the hyperoperation $+$. However, we would like to observe the following fact.

Proposition 3.8. *The operation $*$ defined above on $\mathcal{H}_0(K)$ is not associative.*

Proof. Take $x, y \in K^\times$ with $vx > vy$. By Part 1) of Lemma 3.3 we obtain that $[x + y]_0 = [y]_0$. Therefore,

$$([x]_0 * [y]_0) * [-y]_0 = [x + y]_0 * [-y]_0 = [y]_0 * -[y]_0 = [0]_0.$$

On the other hand,

$$[x]_0 * ([y]_0 * [-y]_0) = [x]_0 * [0]_0 = [x + 0]_0 = [x]_0 \neq [0]_0.$$

Hence,

$$([x]_0 * [y]_0) * [-y]_0 \neq [x]_0 * ([y]_0 * [-y]_0)$$

so the operation $*$ is not associative. □

Remark 3.9. As noted in [50], this operation satisfies the following almost associative law:

$$(a * b) * c \neq a * (b * c) \implies a * b = 0 \text{ or } b * c = 0.$$

With this operation, $\mathcal{H}_0(K)$ forms a *corpoïde* a notion also introduced by Krasner in [23].

If $[b]_0 \neq -[a]_0$, then $[c]_0 \in [a]_0 + [b]_0$ if and only if $[c]_0 = [a]_0 * [b]_0$. On the other hand, $[c]_0 \in [a]_0 - [a]_0$ (for $[a]_0 \neq [0]_0$) if and only if $vc > va$ by Part 2) of Lemma 3.3. By Part 1) of Lemma 3.3 this happens if and only if $[c + a]_0 = [a]_0$. We conclude that $[c]_0 \in [a]_0 - [a]_0$ if and only if $[c]_0 * [a]_0 = [a]_0$. This shows that one can define the hyperoperation $+$ of $\mathcal{H}_0(K)$ using the operation $*$.

We conclude this section looking at $\mathcal{H}_\gamma(K)$ for some $0 < \gamma \in vK$. In this case from Part 2) of Lemma 3.3 we obtain that $[x]_\gamma + [y]_\gamma$ is the singleton containing $[x+y]_\gamma$ when $v(x+y) = \min\{vx, vy\}$. However, the reader should note that it is not true that if $v(x+y) > \min\{vx, vy\}$, then $[0]_\gamma \in [x]_\gamma + [y]_\gamma$. In fact, this is true only if $v(x+y) > \gamma + \min\{vx, vy\}$. If $0 < v(x+y) - \min\{vx, vy\} \leq \gamma$, then $[0]_\gamma \notin [x]_\gamma + [y]_\gamma$ and $[x]_\gamma + [y]_\gamma$ might not be a singleton as the following example shows.

Example 3.10. Consider $\mathbb{Q}$ with the 2-adic valuation v. Let $\gamma = 1$. Then we have $[2]_1 \in [1]_1 + [1]_1$ and $[6]_1 \in [1]_1 + [1]_1$ as $v(6 - (1+1)) = v4 = 2 > 1$. We have that $[2]_1 \neq [6]_1$ since $v(6-2) = 1 + v2$ (here we apply Part 1) of Lemma 3.3). Therefore, $[1]_1 + [1]_1$ is not a singleton. However, $[0]_1 \notin [1]_1 + [1]_1$ follows again by Part 2) of Lemma 3.3, since $v(1+1) = 1$.

Thus, if $\gamma > 0$, then $\mathcal{H}_\gamma(K)$ might be not stringent. In this case, it is unclear how to define an operation on $\mathcal{H}_\gamma(K)$ which would play a similar role as $*$ played for $\mathcal{H}_0(K)$ above. In this case, the notion of hyperfield and the multivalued addition seem to be needed.

3.3 The residue hyperfield of $\mathcal{H}_\gamma(K)$

In this section we show that the residue hyperfield of the γ-valued hyperfield (cf. Definition 2.34) of a valued field (K, v) is always a field isomorphic to the residue field of (K, v). This will follow from some more general results which we establish here and will be useful later in Section 4.2.

Let (K, v) be a valued field and $\gamma \in vK$ be a non-negative element of the value group. For a subset $A \subseteq K$ we set

$$\mathcal{H}_\gamma(A) := \{[x]_\gamma \in \mathcal{H}_\gamma(K) \mid x \in A\}.$$

Lemma 3.11. *Let (K, v) be a valued field and $\gamma \in vK$ be a non-negative element from the value group. Then*

$$\mathcal{O}_{v_\gamma} = \mathcal{H}_\gamma(\mathcal{O}_v).$$

In particular, $\mathcal{H}_\gamma(\mathcal{O}_v)$ is a valuation hyperring in $\mathcal{H}_\gamma(K)$.

Proof. Fix $[x]_\gamma \in \mathcal{H}_\gamma(K)$. By definition, we have that $[x]_\gamma \in \mathcal{O}_{v_\gamma}$ if and only if $v_\gamma[x]_\gamma \geq 0$. This in turn is equivalent to $vx \geq 0$, or to $x \in \mathcal{O}_v$, the latter being equivalent to $[x]_\gamma \in \mathcal{H}_\gamma(\mathcal{O}_v)$.

The last assertion follows by Proposition 2.37. □

Lemma 3.12. *Let (K, v) be a valued field and $\gamma \in vK$ be a non-negative element from the value group. For all $\delta \in vK$ such that $\delta \geq 0$, we have that $\mathcal{H}_\gamma(\mathcal{M}^\delta)$ is a hyperideal of $\mathcal{H}_\gamma(\mathcal{O}_v)$.*

Proof. Let $I := \mathcal{H}_\gamma(\mathcal{M}^\delta)$ and $R := \mathcal{H}_\gamma(\mathcal{O}_v)$. First, note that $[0]_\gamma \in I$. Second, take $[x]_\gamma, [y]_\gamma \in I$ and $[z]_\gamma \in R$. Then $vx, vy > \delta$ and $vz \geq 0$. Assume that $[a]_\gamma \in [x]_\gamma - [y]_\gamma$ so $a = xt - yu$ for some $t, u \in 1 + \mathcal{M}^\gamma$ (Lemma 2.9). We compute

$$va = v(xt - yu) \geq \min\{vx, vy\} > \delta$$

where we used the fact that $vt = vu = 0$. Therefore, $[a]_\gamma \in I$ and we have proved that $[x]_\gamma - [y]_\gamma \subseteq I$. Further, $[x]_\gamma[z]_\gamma = [xz]_\gamma$ and

$$v(xz) = vx + vz > \delta.$$

Hence, $[x]_\gamma[z]_\gamma \in I$ and I is a hyperideal of R. □

Lemma 3.13. *Let (K, v) be a valued field and $\gamma \in vK$ be a non-negative element from the value group. Denote by R the valuation hyperring $\mathcal{H}_\gamma(\mathcal{O}_v)$ of $\mathcal{H}_\gamma(K)$ and by I its hyperideal $\mathcal{H}_\gamma(\mathcal{M}^\delta)$ where $0 \leq \delta \leq \gamma$. Then for all $a, b \in R/I$ we have that $a + b$ is a singleton.*

For $x \in \mathcal{O}_v$, the corresponding element of R/I should be denoted by $[[x]_\gamma]_I$. We will instead denote it by $[x]_{\gamma,I}$ to simplify the notation.

Proof. By definition of quotient hyperring modulo a hyperideal, we have

$$[x]_{\gamma,I} + [y]_{\gamma,I} = \{[z]_{\gamma,I} \mid [z]_\gamma \in [x]_\gamma + [y]_\gamma\}$$

for all $[x]_{\gamma,I}, [y]_{\gamma,I} \in R/I$. By Lemma 2.9 we conclude that if $[z]_{\gamma,I} \in [x]_{\gamma,I} + [y]_{\gamma,I}$, then we can write $z = xt + yu$ for some $t, u \in 1 + \mathcal{M}^\gamma$.

Fix $[x]_{\gamma,I}$, $[y]_{\gamma,I} \in R/I$ and take

$$[z]_{\gamma,I}\ , [z']_{\gamma,I} \in [x]_{\gamma,I} + [y]_{\gamma,I}\ .$$

We wish to show that $[z]_{\gamma,I} = [z']_{\gamma,I}$ or, in other words, that $[z]_\gamma \sim_I [z']_\gamma$. We claim that $[z - z']_\gamma \in ([z]_\gamma - [z]_\gamma) \cap I$. Write $z = xt + yu$ and $z' = xt' + yu'$ for $t, u, t', u' \in 1 + \mathcal{M}^\gamma$. We have $z - z' = x(t - t') + y(u - u')$ and then

$$v(z - z') \geq \min\{v(t - t') + vx, v(u - u') + vy\} \geq \min\{v(t - t'), v(u - u')\} \quad (3.2)$$

We now show that $v(t - t') > \delta$. Write $t = 1 + d$ and $t' = 1 + d'$ for $d, d' \in \mathcal{M}^\gamma$, we have

$$v(t - t') = v(1 + d - 1 - d') = v(d - d') \geq \min\{vd, vd'\} > \gamma \geq \delta.$$

Similarly, one can show that $v(u-u') > \delta$. From (3.2), we deduce that $v(z-z') > \delta$, whence $[z-z']_\gamma \in I$. On the other hand, $[z-z']_\gamma \in [z]_\gamma - [z']_\gamma$ by Part 4) of Lemma 3.3. This completes the proof. □

The condition $\delta \le \gamma$ is needed, as the next example shows.

Example 3.14. Let us consider $\mathbb{Q}(t)$ with the t-adic valuation v. Let $\gamma = 0$ and take $\delta = 1$. Denote by I the hyperideal $\mathcal{H}_0(\mathcal{M}^1)$ of $R := \mathcal{H}_0(\mathcal{O}_v)$. We consider

$$[1]_{0,I} - [1]_{0,I} = \{[f(t)]_{0,I} \mid [f(t)]_0 \in [1]_0 - [1]_0\}.$$

By Part 2) of Lemma 3.3 we have that $[f(t)]_0 \in [1]_0 - [1]_0$ if and only if $vf(t) > 0$. Hence,

$$[t]_{0,I}\ , [t^2]_{0,I} \in [1]_{0,I} - [1]_{0,I}.$$

However, $[t]_{0,I} \neq [t^2]_{0,I}$ since

$$([t]_0 - [t^2]_0) \cap I = \emptyset,$$

as the following argument shows. Using Part 2) of Lemma 3.3, we have that $[f(t)]_0 \in [t]_0 - [t^2]_0$ if and only if $v(f(t) - t + t^2) > vt = 1$. Now, assume that $vf(t) > 1$, i.e., $f(t) \in I$. Then

$$v(f(t) - t + t^2) = \min\{v(f(t) + t^2), vt\} = 1,$$

so $f(t) \notin [t]_0 - [t^2]_0$. We have shown that $[1]_{0,I} - [1]_{0,I}$ is not a singleton.

Following what we have said in Remark 2.4 we can restate Lemma 3.13 by saying that $\mathcal{H}_\gamma(\mathcal{O}_v)/\mathcal{H}_\gamma(\mathcal{M}^\delta)$ is a ring whenever $0 \le \delta \le \gamma$.

The next result shows that, in this case, $\mathcal{H}_\gamma(\mathcal{O}_v)/\mathcal{H}_\gamma(\mathcal{M}^\delta)$ is isomorphic to the δ-residue ring $\mathcal{O}_v^\delta = \mathcal{O}^\delta := \mathcal{O}_v/\mathcal{M}^\delta$ of (K, v).

Proposition 3.15. *Let (K, v) be a valued field and $\gamma \in vK$ be a non-negative element from the value group. Denote by R the valuation hyperring $\mathcal{H}_\gamma(\mathcal{O}_v)$ of $\mathcal{H}_\gamma(K)$ and by I its hyperideal $\mathcal{H}_\gamma(\mathcal{M}^\delta)$ where $0 \le \delta \le \gamma$. The map*

$$\begin{aligned}\sigma = \sigma_{v,\delta} : R/I &\to \mathcal{O}^\delta \\ [x]_{\gamma,I} &\mapsto x + \mathcal{M}^\delta\end{aligned}$$

is an isomorphism of hyperrings.

Proof. We first have to show that σ is well-defined. To this end, take $[x]_\gamma, [y]_\gamma \in R$ such that $[x]_{\gamma,I} = [y]_{\gamma,I}$. This means that $([x]_\gamma - [y]_\gamma) \cap I \neq \emptyset$. We wish to show that $x - y \in \mathcal{M}^\delta$. Let $[z]_\gamma \in ([x]_\gamma - [y]_\gamma) \cap I$ and write $z = x(1 + d) - y(1 + d')$ for some $d, d' \in \mathcal{M}^\gamma$ (Lemma 2.9). Thus, $x - y = z - xm + ym$ and

$$v(x - y) = v(z - xm + ym') \ge \min\{vz, vx + vm, vy + vm'\} > \delta.$$

To see that σ is surjective we take $x + \mathcal{M}^\delta \in \mathcal{O}^\delta$. In particular, $x \in \mathcal{O}_v$, whence $[x]_\gamma \in R$ and $[x]_{\gamma,I} \in R/I$ is such that $\sigma([x]_{\gamma,I}) = x + \mathcal{M}^\delta$.

To see that σ is injective we assume that $x-y \in \mathcal{M}^\delta$. We wish to show that $[x]_{\gamma,I} = [y]_{\gamma,I}$, or in other words $([x]_\gamma - [y]_\gamma) \cap I \neq \emptyset$. By Part 4) of Lemma 3.3 we have that $[x-y]_\gamma \in [x]_\gamma - [y]_\gamma$, on the other hand by assumption $[x-y]_\gamma \in I$, since $x-y \in \mathcal{M}^\delta$. Thus, $([x]_\gamma - [y]_\gamma) \cap I \neq \emptyset$ as we wished to show.

We interpret the ring $\mathcal{O}^\delta$ as a hyperring by setting

$$(x+\mathcal{M}^\delta) + (y+\mathcal{M}^\delta) := \{x+y+\mathcal{M}^\delta\}$$

for all $x, y \in \mathcal{O}_v$ (cf. Remark 2.4). In order to show that σ is a strict homomorphism of hyperrings we recall that, by Lemma 3.13, the subset $[x]_{\gamma,I} + [y]_{\gamma,I}$ of R/I is a singleton for all $[x]_{\gamma,I}$, $[y]_{\gamma,I} \in R/I$. Since by Part 4) of Lemma 3.3 we have that $[x+y]_\gamma \in [x]_\gamma + [y]_\gamma$, we conclude that

$$[x]_{\gamma,I} + [y]_{\gamma,I} = \{[x+y]_{\gamma,I}\}.$$

for all $[x]_{\gamma,I}$, $[y]_{\gamma,I} \in R/I$. Therefore,

$$\begin{aligned}\sigma([x]_{\gamma,I} + [y]_{\gamma,I}) &= \sigma(\{[x+y]_{\gamma,I}\}) \\ &= \{\sigma([x+y]_{\gamma,I})\} \\ &= \{x+y+\mathcal{M}^\delta\} \\ &= (x+\mathcal{M}^\delta) + (y+\mathcal{M}^\delta) \\ &= \sigma([x]_{\gamma,I}) + \sigma([y]_{\gamma,I}).\end{aligned}$$

Regarding the multiplication we have:

$$\begin{aligned}\sigma([x]_{\gamma,I}[y]_{\gamma,I}) &= \sigma([xy]_{\gamma,I}) \\ &= xy+\mathcal{M}^\delta \\ &= (x+\mathcal{M}^\delta)(y+\mathcal{M}^\delta) \\ &= \sigma([x]_{\gamma,I})\sigma([y]_{\gamma,I}).\end{aligned}$$

This completes the proof. □

Taking $\delta = 0$ we obtain the promised result on the residue of the γ-valued hyperfield, which ends this section.

Corollary 3.16. *Let (K, v) be a valued field and $\gamma \in vK$ be a non-negative element from the value group. The residue hyperfield of $(\mathcal{H}_\gamma(K), v_\gamma)$ is a field isomorphic to Kv.*

Before ending this section, let us observe that the 0-valued hyperfield of a valued field (K, v) always has a subhyperring isomorphic to Kv.

Proposition 3.17. *Let (K, v) be a valued field. Then $F := \mathcal{H}_0(\mathcal{O}_v^\times) \cup \{[0]_0\}$ is a subhyperring of $\mathcal{H}_0(K)$ which is a hyperfield and in which the sum of two elements is always a singleton. Moreover, F is isomorphic to Kv.*

Proof. Clearly $[0]_0, [1]_0 \in F$ and $[xy]_0 \in F$ for all $[x]_0, [y]_0 \in F$. Further, $F^\times$ is a group. We will now show that the induced hyperoperation

$$[x]_0 +_F [y]_0 := ([x]_0 + [y]_0) \cap F \quad ([x]_0, [y]_0 \in F)$$

always results in a singleton. We distinguish two cases:

- if $[0]_0 \notin [x]_0 + [y]_0$, then by Part 2) of Lemma 3.3 we have that $v(x + y) = 0$. Thus $[x + y]_0 \in F$ and we have that $[x]_0 +_F [y]_0 = \{[x + y]_0\}$ by Lemma 3.6.
- if $[0]_0 \in [x]_0 + [y]_0$, then $[y]_0 = -[x]_0$ and we have that $[z]_0 \in [x]_0 - [x]_0$ if and only if $vz > 0$ by Part 2) of Lemma 3.3. This shows that $[x]_0 +_F [y]_0 = \{[0]_0\}$ in this case.

It is now clear that F forms a (hyper)field and that the map $F \ni [x]_0 \mapsto xv \in Kv$ defines an isomorphism. □

For $\gamma > 0$ the same might not work.

Example 3.18. Let (K, v) be $\mathbb{Q}$ with the 2-adic valuation and $\gamma := 1$. Set $F := \mathcal{H}_1(\mathcal{O}_v^\times) \cup \{0\}$. Then $[1]_1 +_F [1]_1 = \emptyset$. Indeed, $[0]_1 \notin [1]_1 + [1]_1$ by Part 2) of Lemma 3.3, since $v2 = 1$. Moreover, if $vx = 0$, then $v(x - 2) = 0 < 1$ and therefore $[x]_0 \notin [1]_0 + [1]_0$ again by Part 2) of Lemma 3.3.

3.4 Properties of the γ-valued hyperfield

In his article [32], Lee already introduced the notion of the γ-valued hyperfield. However, in that paper the definition of valued hyperfield is different from the one of Davvaz and Salasi which we use. Specifically, two more properties are required (cf. [32, Definition 2.4]). In his paper Lee does not show explicitly that the γ-valued hyperfields satisfy these properties. In this section we wish to clarify this issue providing more details. The two properties mentioned above will be stated in Proposition 3.19 and Proposition 3.26 below. It is worth mentioning that the main tool used in the proofs of both of these results is Lemma 3.3 which is proved in the paper of Lee.

Proposition 3.19. *Let $(\mathcal{H}_\gamma(K), v_\gamma)$ be the γ-valued hyperfield of a valued field (K, v), where $0 \leq \gamma \in vK$. Then for all $[x]_\gamma$, $[y]_\gamma \in \mathcal{H}_\gamma(K)$ we have that $v_\gamma([x]_\gamma + [y]_\gamma)$ consists of a single element, unless $[0]_\gamma \in [x]_\gamma + [y]_\gamma$.*

Proof. Let $[x]_\gamma, [y]_\gamma \in \mathcal{H}_\gamma(K)$ be given such that $[0]_\gamma \notin [x]_\gamma + [y]_\gamma$. We have to show that v_γ is constant on $[x]_\gamma + [y]_\gamma$. Assume without loss of generality that $vx \le vy$ and take $[z]_\gamma, [z']_\gamma \in [x]_\gamma + [y]_\gamma$. Since $[0]_\gamma \notin [x]_\gamma + [y]_\gamma$, by Part 3) of Lemma 3.3 there exists $[a]_\gamma \in [x]_\gamma + [y]_\gamma$ such that $va \le \gamma + vx$. By Part 2) of Lemma 3.3 we have that

$$v(z - (x + y)) > \gamma + vx$$

and

$$v(a - (x + y)) > \gamma + vx.$$

We compute

$$v(z-a) = v(z-(x+y)+(x+y)-a) \ge \min\{v(z-(x+y)), v(a-(x+y))\} > \gamma+vx \ge va.$$

Therefore,

$$vz = v(z - a + a) = \min\{v(z - a), va\} = va.$$

Similarly, we obtain $vz' = va$. Thus, $v_\gamma[z]_\gamma = v_\gamma[z']_\gamma$ as required. □

For a better understanding of the second property of γ-valued hyperfields to which we referred above, we will now introduce some concepts from the theory of ultrametric spaces.

Definition 3.20. An *ultrametric* on a set X is a function $d : X \times X \to \Gamma \cup \{\infty\}$, where $(\Gamma, <)$ is a linearly ordered set and ∞ satisfies $\gamma < \infty$ for all $\gamma \in \Gamma$, such that for all $x, y, z \in X$

(U1) $d(x, y) = \infty$ if and only if $x = y$,

(U2) $d(x, y) = d(y, x)$,

(U3) $d(x, z) \ge \min\{d(x, y), d(y, z)\}$.

We call (X, d) an *ultrametric space* if d is an ultrametric on X. We introduce the following notation $dX := \{d(x, y) \mid x, y \in X \wedge x \neq y\} \subseteq \Gamma$ and call dX the *value set* of d.

Definition 3.21. Let (X, d) be an ultrametric space. A subset $B \subseteq X$ is called a *ball* if

$$\forall y, z \in B\ \forall w \in X (d(y, w) \ge d(y, z) \to w \in B)$$

For every $x \in X$ and every final segment S of $dX \cup \{\infty\}$, we define

$$B_S(x) := \{y \in X \mid d(x, y) \in S\}.$$

Lemma 3.22. *Every set $B_S(x)$ is a ball in X. Conversely, if B is a ball in X and S is the least final segment containing the elements $d(y,z)$ for all $y,z \in B$, then for every $x \in B$,*

$$B = B_S(x).$$

In particular, $B_S(x) = B_S(y)$ for every $y \in B_S(x)$.

Proof. Assume that $y,z \in B_S(x)$, that is, $d(x,y) \in S$ and $d(x,z) \in S$. Suppose in addition that $w \in X$ is such that $d(y,w) \geq d(y,z)$. Then $d(x,w) \geq \min\{d(x,y), d(y,w)\} \geq \min\{d(x,y), d(y,z)\} \geq \min\{d(x,y), d(y,x), d(x,z)\} \in S$. Since S is a final segment of $dX \cup \{\infty\}$, it follows that $d(x,w) \in S$. Hence $w \in B_S(x)$. This shows that $B_S(x)$ is a ball.

For the converse, assume that B is a ball and let S be as in the assertion. Further, let x be any element in B. If $y \in B$, then $d(x,y) \in S$ and thus $y \in B_S(x)$. On the other hand, if $y \in B_S(x)$, then $d(x,y) \in S$. So by definition of S, there is some $z \in B$ such that $d(x,z) \leq d(x,y)$. Since B is a ball, it follows that $y \in B$. We have proved that $B = B_S(x)$. □

Definition 3.23. Let (X,d) be an ultrametric space, $x \in X$ and $\alpha \in dX$. The *closed ball of radius α around x* is

$$B_\alpha(x) := \{y \in X \mid d(x,y) \geq \alpha\}.$$

The *open ball of radius α around x* is

$$B^o_\alpha(x) := \{y \in X \mid d(x,y) > \alpha\}.$$

Thus, $B^o_\alpha(x) = B_S(x)$ where $S = \{\gamma \in dX \cup \{\infty\} \mid \gamma > \alpha\}$.

Lemma 3.24. *Let (X,d) be an ultrametric space. Every two balls with non-empty intersection are comparable by inclusion. In particular, for all $\alpha, \beta \in dX$ and all $x,y \in X$,*

$$\alpha \geq \beta \wedge B^o_\alpha(x) \cap B^o_\beta(y) \neq \emptyset \Longrightarrow B^o_\alpha(x) \subseteq B^o_\beta(y).$$

Proof. Take two balls B and B' and suppose that $z \in B \cap B'$. By Lemma 3.22 there are final segments of $dX \cup \{\infty\}$ such that $B = B_S(z)$ and $B' = B_{S'}(z)$. Since S and S' are final segments we must have $S \subseteq S'$ or $S' \subseteq S$. Hence $B \subseteq B'$ or $B' \subseteq B$.

For the second assertion we just have to note that $\alpha \geq \beta$ implies that

$$\{\gamma \in dX \cup \{\infty\} \mid \gamma > \alpha\} \subseteq \{\gamma \in dX \cup \{\infty\} \mid \gamma > \beta\}.$$ □

Proposition 3.19 allows us to define the following ultrametric on $\mathcal{H}_\gamma(K)$:

$$d([x]_\gamma,[y]_\gamma) := \begin{cases} v(x-y) & \text{if } [y]_\gamma \neq [x]_\gamma \\ \infty & \text{otherwise} \end{cases}$$

To show that d is well-defined, take $x' \in [x]_\gamma$ and $y' \in [y]_\gamma$. Then $[x'-y']_\gamma \in [x]_\gamma - [y]_\gamma$. On the other hand, by Part 4) of Lemma 3.3 $[x-y]_\gamma \in [x]_\gamma - [y]_\gamma$ too. Thus, if $[y]_\gamma \neq [x]_\gamma$ we have that $v(x'-y') = v(x-y)$ by Proposition 3.19 and so $d([x']_\gamma,[y']_\gamma) = d([x]_\gamma,[y]_\gamma)$. If $[x]_\gamma = [y]_\gamma$, then $[x']_\gamma = [x]_\gamma = [y]_\gamma = [y']_\gamma$, whence $d([x']_\gamma,[y']_\gamma) = \infty = d([x]_\gamma,[y]_\gamma)$. Properties (U1), (U2) and (U3) are straightforward to verify: they follow from the corresponding properties of v.

Remark 3.25. We remark that defining $d([x]_\gamma,[y]_\gamma) := v_\gamma([x]_\gamma - [y]_\gamma)$ for $[x]_\gamma \neq -[y]_\gamma$, would not yield an ultrametric with value set vK as $v_\gamma([x]_\gamma - [y]_\gamma)$ is a singleton and as such an element of $\mathcal{P}^*(vK)$. However, we know from Part 4) of Lemma 3.3 that the only element of this singleton is $v_\gamma[x-y]_\gamma = v(x-y)$. This justifies our choice.

Proposition 3.26. *Let $(\mathcal{H}_\gamma(K), v_\gamma)$ be the γ-valued hyperfield of a valued field (K,v), for some $0 \leq \gamma \in vK$. If x and y are not both 0, then $[x]_\gamma + [y]_\gamma$ is the open ball of radius $\delta := \gamma + \min\{vx, vy\}$ around $[x+y]_\gamma$ with respect to the ultrametric d defined above on $\mathcal{H}_\gamma(K)$.*

Proof. Take $[x]_\gamma, [y]_\gamma \in \mathcal{H}_\gamma(K)$. If $[z]_\gamma \in [x]_\gamma + [y]_\gamma$ is such that $[z]_\gamma \neq [x+y]_\gamma$, then by Part 2) of Lemma 3.3 we have that

$$d([z]_\gamma,[x+y]_\gamma) = v(z-(x+y)) > \delta.$$

On the other hand, $d([x+y]_\gamma,[x+y]_\gamma) = \infty > \delta$. This shows that for every $[z]_\gamma \in [x]_\gamma + [y]_\gamma$ we have $[z]_\gamma \in B^o_\delta([x+y]_\gamma)$, hence $[x]_\gamma + [y]_\gamma \subseteq B^o_\delta([x+y]_\gamma)$.

For the other inclusion, take $[z]_\gamma \in B^o_\delta([x+y]_\gamma)$. Thus,

$$d([z]_\gamma,[x+y]_\gamma) > \delta.$$

If $[z]_\gamma = [x+y]_\gamma$, then $[z]_\gamma \in [x]_\gamma + [y]_\gamma$ by Part 4) of Lemma 3.3. If $[z]_\gamma \neq [x+y]_\gamma$, then

$$v(z-(x+y)) = d([z]_\gamma,[x+y]_\gamma) > \delta$$

therefore, by Part 2) of Lemma 3.3 we conclude that $[z]_\gamma \in [x]_\gamma + [y]_\gamma$. □

Remark 3.27. Let (F,v) be a (not necessarily discrete) valued hyperfield such that $v(x+y)$ is a singleton $\{\gamma_{x,y}\}$ unless $0 \in x+y$ and assume that there is $\rho_F \in vF$

such that for all $x, y \in F$ we have that $x + y$ is an open or a closed ultrametric ball of radius $\rho_F + \min\{vx, vy\}$ with respect to the ultrametric

$$d(x,y) := \begin{cases} \gamma_{x,y} & \text{if } v(x-y) = \{\gamma_{x,y}\} \\ \infty & \text{otherwise.} \end{cases}$$

In other words, let (F, v) be a valued hyperfield in the sense of [32, Definition 2.4]. Observe that $\mathcal{M}_v = B_0^o(0)$ is an ultrametric ball. In particular, $(x-y) \cap \mathcal{M}_v \neq \emptyset$ if and only if $x - y \subseteq \mathcal{M}_v$ or $\mathcal{M}_v \subseteq x - y$ for all $x, y \in \mathcal{O}_v$ by Lemma 3.24. Note that if $\mathcal{M}_v \subseteq x - y$, then $x = y$ must hold, as $0 \in \mathcal{M}_v$. Suppose that $z \in (x - x) \setminus \mathcal{M}_v$. Then $vz = 0$ by Lemma 2.32 and then $0 = vz \geq vx \geq 0$ by (V3) showing that $vx = 0$ too. Now, since $0 \in x - x$, it is a center of the ball $x - x$, thus $z \in x - x$ implies $0 = vz \geq \rho_F + vx = \rho_F$. Therefore, if $\rho_F > 0$, then $(x - y) \cap \mathcal{M}_v \neq \emptyset$ if and only if $x - y \subseteq \mathcal{M}_v$, since the converse inclusion cannot occur.

If, as in the case of $\mathcal{H}_\gamma(K)$, the ultrametric ball resulting as the sum of two elements is always open, then also for $\rho_F = 0$ we have $(x - y) \cap \mathcal{M}_v \neq \emptyset$ if and only if $x - y \subseteq \mathcal{M}_v$. Thus, in these cases our notion of residue hyperfield and the notion of residue field given in [32, Definition 2.20] coincide. Indeed, the relation $x \equiv_\theta y$ defined in [32, Definition 2.20] and thereby used to define the residue is equivalent to $x - y \subseteq \mathcal{M}_v$.

It follows that in these cases the residue hyperfield Fv is a field. To see this take $[x]_{\mathcal{M}_v}, [y]_{\mathcal{M}_v} \in Fv$ and consider $[z]_{\mathcal{M}_v}, [z']_{\mathcal{M}_v} \in [x]_{\mathcal{M}_v} + [y]_{\mathcal{M}_v}$. If $0 \in z - z'$, then $z = z'$ and there is nothing to show. Otherwise since

$$z, z' \in x + y \subseteq B_{\rho_F + \min\{vx, vy\}}(z),$$

we have $\gamma_{z,z'} \geq \rho_F + \min\{vx, vy\} \geq \rho_F$. Hence, if $\rho_F > 0$, then $\gamma_{z,z'} \in (z - z') \cap \mathcal{M}_v$ and so $[z]_{\mathcal{M}_v} = [z']_{\mathcal{M}_v}$. If $\rho_F = 0$ and $x + y = B^o_{\rho_F + \min\{vx, vy\}}(z)$, then we obtain $\gamma_{z,z'} > \min\{vx, vy\} \geq 0$, so once again $\gamma_{z,z'} \in (z - z') \cap \mathcal{M}_v$ and $[z]_{\mathcal{M}_v} = [z']_{\mathcal{M}_v}$ follows. Thus, in these cases $[x]_{\mathcal{M}_v} + [y]_{\mathcal{M}_v}$ is a singleton.

Let us now analyze an example of a valued hyperfield in our sense which is not a valued hyperfield in original sense of Krasner (see e.g. [32, Definition 2.4]).

Example 3.28. Let $K := \mathbb{R}(X)$ and v be the X-adic valuation on K, defined by $v(X) = 1$. Note that this yields a discrete valued field (K, v). Consider $T := \mathbb{R}^\times$, the subgroup of $K^\times$ consisting of all nonzero real numbers. Clearly, $T \subseteq \mathcal{O}_v^\times$ and thus on the factor hyperfield $F := K_T$ we have a valuation v_T .

Since $\frac{-1}{X-1} \notin T$ we have that $-[1]_T \neq [X-1]_T$. Therefore, $[0]_T \notin [1]_T + [X-1]_T$. However, we have that $[X]_T, [X-2]_T \in [1]_T + [X-1]_T$ and

$$v_T[X]_T = 1 \neq 0 = v_T[X-2]_T.$$

Hence, $v_T([1]_T + [X-1]_T)$ is not a singleton.

At this point one should make clear what "ball" means in the last axiom of [32, Definition 2.4], i.e., what ultrametric has to be considered. Assume that for all $x, y \in K$ such that $[0]_T \notin [x]_T - [y]_T$ we have chosen an element $[z_{x,y}]_T \in [x]_T - [y]_T$ and set $d([x]_T, [y]_T) := v_T[z_{x,y}]_T$. Assume further that setting $d([x]_T, [x]_T) := \infty$ we obtain an ultrametric on F with value set vK. Restricting our attention to this kind of ultrametrics, we can observe that $[1]_T - [1]_T = \{[0]_T, [1]_T\}$ cannot be a closed or an open ball. Indeed, suppose that $[x]_T \in F$ and $\gamma \in vK$ are such that $B^o_\gamma([x]_T) = \{[0]_T, [1]_T\}$. Then $[x]_T = [0]_T$ or $[x]_T = [1]_T$. However, since any element of an ultrametric ball is a center, we can assume that $[x]_T = [0]_T$. In this case, $v_T[1]_T = v1 = 0 > \gamma$ since $[1]_T \in B_\gamma([0]_T)$. Now, we have that

$$[X]_T - [0]_T = \{[X]_T\}$$

and $v_T[X]_T = 1 > 0 > \gamma$, hence $[X]_T \in B^o_\gamma([0]_T) = \{[0]_T, [1]_T\}$, a contradiction since $\frac{1}{X} \notin T$ and $[X]_T \neq [0]_T$. This shows that $[1]_T - [1]_T$ cannot be an open ball. The argument for closed balls is similar.

Finally, let us show that the residue hyperfield of this valued hyperfield is not a field. Denote by $\mathcal{M}$ the maximal hyperideal $\mathcal{M}_{v_T}$. By definition of the hyperaddition in the residue hyperfield and Lemma 2.9, we have that

$$[1]_{T,\mathcal{M}} + [1]_{T,\mathcal{M}} = \{[t+u]_{T,\mathcal{M}} \mid t, u \in T\} = \{[0]_{T,\mathcal{M}}, [1]_{T,\mathcal{M}}\}.$$

Now, $[0]_{T,\mathcal{M}} \neq [1]_{T,\mathcal{M}}$ since $v_T[1]_T = v1 = 0$ so that $[1]_T \notin \mathcal{M}$. However, $[1]_T$ is the only element of $[1]_T - [0]_T$, so $([1]_T - [0]_T) \cap \mathcal{M} = \emptyset$. This shows that Fv_T is not a field.

We can learn from the last example that the more general definition of valuation given by Davvaz and Salasi that we adopted covers some interesting cases which are left out by the original definition of valued hyperfield used for instance by Tolliver [48, 49] and Lee [32]. However, adopting the more general definition one has to be prepared for the fact that the residue might not be a field. This yields for instance to the problem of defining the residue characteristic of a valued hyperfield. In the literature one may find two distinct notions for the characteristic of a hyperfield (see [53, Section 4.6]).

A remarkable application of Proposition 3.26 is the following characterization of the subhyperrings of $\mathcal{H}_\gamma(K)$. This result was proved by Tolliver in [49, Proposition 5.1.3] for valuations with values in $\mathbb{R}$.

Theorem 3.29. *Take a valued field (K, v) and $\gamma \in vK$ be a non-negative element of the value group. Let $F := \mathcal{H}_\gamma(K)$ and $H \subseteq F$. Then H is a subhyperring of F if and only if $0_F \in H$ and for all $x, y \in H$ we have $xy \in H$, $-x \in H$ and $(x+y) \cap H \neq \emptyset$.*

Proof. If H is a subhyperring of F, then $x+_H y := (x+y)\cap H$ gives to $(H, +_H, \cdot, 0_F)$ the structure of a hyperring. In particular, $(x+y)\cap H \neq \emptyset$ and $-x, xy \in H$ for all $x, y \in H$ and $0_F \in H$. Note that, by the uniqueness required in (H3), for all $x \in H$ the (additive) inverse of x in H must be the same as its inverse in F, namely, $-x$.

Conversely, we have to show that $(H, +_H, \cdot, 0_F)$ is a hyperring. We wish to show associativity of $+_H$. Let $x, y, z \in H$ and take

$$w \in (x +_H y) +_H z = (((x + y) \cap H) + z) \cap H.$$

Since $w \in x + (y + z)$, we have that there exists $b \in y + z$ such that $w \in x + b$. Therefore, by the reversibility axiom we obtain that $b \in w - x$. Thus,

$$b \in (w - x) \cap (y + z) \neq \emptyset.$$

By Proposition 3.26 we have that $w - x$ and $y + z$ are ultrametric balls (or the singleton $\{0_F\}$), so, since they are not disjoint, one must be contained in the other by Lemma 3.24. Therefore, $(w-x)\cap(y+z)\cap H$ is either $(w-x)\cap H$ or $(y+z)\cap H$ and in both cases it is non-empty by assumption. Take $a \in (w - x) \cap (y + z) \cap H$. By $a \in w-x$ we obtain that $w \in x+a$ using the reversibility axiom. Since $w \in H$, we have $w \in (x + a) \cap H$ and by $a \in (y + z) \cap H$ we derive

$$(x + a) \cap H \subseteq (x + ((y + z) \cap H)) \cap H.$$

Hence, $w \in x +_H (y +_H z)$ and then $(x +_H y) +_H z \subseteq x +_H (y +_H z)$. The converse inclusion is similar.

The other axioms follow directly from the corresponding axioms valid in F. □

Remark 3.30. A straightforward expression of the associativity axiom for hyperrings is given by the following $\mathcal{L}_{hr}$-formula

$$\forall x \forall y \forall z \forall t ((\exists a (r_+(x, y, a) \wedge r_+(a, z, t))) \leftrightarrow (\exists b (r_+(y, z, b) \wedge r_+(x, b, t))))$$

This is not a universal formula as it can be easily checked by putting it in prenex normal form. Therefore, at this stage one cannot ensure that associativity is automatically inherited by $\mathcal{L}_{hr}$-substructures. Theorem 3.29 above, indicates that in the case of the γ-valued hyperfields this problem does not occur. In Appendix B we in fact give a way to write a universal axiom for associativity which works in the case of the γ-valued hyperfields.

In order to apply the above theorem let us consider an extension $(L, w)|(K, v)$ of valued fields and pick a non-negative $\gamma \in vK$. Let us point out that $\mathcal{H}_\gamma(K)$ is not a subset of $\mathcal{H}_\gamma(L)$. Nevertheless, we have an injective map

$$\phi : \mathcal{H}_\gamma(K) \to \mathcal{H}_\gamma(L)$$

which is defined as $\phi(x(1+\mathcal{M}_v^\gamma)) := x(1+\mathcal{M}_w^\gamma)$. This is well-defined since $\mathcal{M}_v^\gamma \subseteq \mathcal{M}_w^\gamma$. To see that this map is injective, assume that $x(1+\mathcal{M}_w^\gamma) = y(1+\mathcal{M}_w^\gamma)$ for some $x, y \in K^\times$ (if $x = 0$ or $y = 0$ there is nothing to show). Then $xy^{-1} \in 1 + \mathcal{M}_w^\gamma$. Now since w is an extension of v, $\gamma \in vK$ and $x, y \in K$ we obtain that $xy^{-1} \in 1 + \mathcal{M}_v^\gamma$ must hold. Therefore, $x(1+\mathcal{M}_v^\gamma) = y(1+\mathcal{M}_v^\gamma)$ and ϕ is injective.

Corollary 3.31. *With the notation introduced above, we have that $\phi(\mathcal{H}_\gamma(K))$ is a subhyperring of $\mathcal{H}_\gamma(L)$.*

Proof. During this proof we will denote the elements of $\mathcal{H}_\gamma(K)$ by $[x]_K$ the elements of $\mathcal{H}_\gamma(L)$ by $[x]_L$. We want to apply Theorem 3.29 with $F = \mathcal{H}_\gamma(L)$ and

$$H = \phi(\mathcal{H}_\gamma(K)) = \{[x]_L \mid x \in K\}$$

The only requirment which is not trivial to verify is the last one, i.e., we have to check that for all $[x]_L$, $[y]_L \in \phi(\mathcal{H}_\gamma(K))$ we have that

$$([x]_L +_L [y]_L) \cap \phi(\mathcal{H}_\gamma(K)) \neq \emptyset$$

where $+_L$ is the hyperoperation of $\mathcal{H}_\gamma(L)$. We compute

$$\begin{aligned}([x]_L +_L [y]_L) \cap \phi(\mathcal{H}_\gamma(K)) &= \{[x+yt]_L \mid t \in 1+\mathcal{M}_w^\gamma\} \cap \phi(\mathcal{H}_\gamma(K)) \\ &= \{[x+yt]_L \mid t \in 1+\mathcal{M}_w^\gamma, x+yt \in K\} \\ &= \{[x+yt]_L \mid t \in 1+\mathcal{M}_v^\gamma\} \\ &= \phi([x]_K +_K [y]_K) \neq \emptyset\end{aligned}$$

where $+_K$ is the hyperoperation of $\mathcal{H}_\gamma(K)$ and we have used the fact that $x, y \in K$ and that $\mathcal{M}_v^\gamma = K \cap \mathcal{M}_w^\gamma$. This completes the proof. □

Remark 3.32. Note that in the above proof we have shown that ϕ is a strict (in our sense, cf. Definition 2.11) homomorphism of hyperrings. In model theoretic terms ϕ is then an embedding (cf. Definition 1.6) in the language of hyperrings.

Remark 3.33. In what follows we identify $\mathcal{H}_\gamma(K)$ with its image under ϕ. With this identification, $\mathcal{H}_\gamma(K)$ is a substructure of $\mathcal{H}_\gamma(L)$ in the language of hyperrings (cf. Remark 1.7). Moreover, since $[x]_\gamma^{-1} = [x^{-1}]_\gamma$, $\mathcal{H}_\gamma(K)$ is a substructure of $\mathcal{H}_\gamma(L)$ in the language of hyperfields too. We actually have that $(\mathcal{H}_\gamma(K), v_\gamma)$ is a substructure of $(\mathcal{H}_\gamma(L), w_\gamma)$ in the language of valued hyperfields. To see this one has to check how the valuation hyperrings behave. That is, one has to verify that

$$\mathcal{O}_{v_\gamma} = \mathcal{O}_{w_\gamma} \cap \mathcal{H}_\gamma(K).$$

This is easily seen once we recall (cf. Lemma 3.11) that

$$\mathcal{O}_{w_\gamma} = \mathcal{H}_\gamma(\mathcal{O}_w) = \{[x]_\gamma \mid x \in \mathcal{O}_w\}$$

which, since $\mathcal{O}_w \cap K = \mathcal{O}_v$, implies that

$$\mathcal{O}_{w_\gamma} \cap \mathcal{H}_\gamma(K) = \{[x]_\gamma \mid x \in \mathcal{O}_v\} = \mathcal{O}_{v_\gamma}.$$

3.5 The case of complete valued fields

In the sequel, by *homomorphism of hyperfields* we mean a homomorphism of hyperrings which is a homomorphism of the multiplicative groups of non-zero elements. The following lemma shows that the γ-valued hyperfields of some valued field (K, v), for $\gamma \in vK_{\geq 0}$, form a projective system.

Lemma 3.34. *Let (K, v) be a valued field and $0 \leq \gamma \leq \delta$ be elements of vK. Then*

$$\begin{aligned} \rho_{\delta,\gamma} : \mathcal{H}_\delta(K) &\to \mathcal{H}_\gamma(K) \\ [x]_\delta &\mapsto [x]_\gamma \end{aligned}$$

is a surjective homomorphism of hyperfields such that $v_\delta[x]_\delta = v_\gamma(\rho_{\delta,\gamma}[x]_\delta)$ for all $[x]_\delta \in \mathcal{H}_\delta(K)$. Furthermore, if $0 \leq \alpha \leq \gamma \leq \delta$ are elements of vK, then

$$\rho_{\delta,\alpha} = \rho_{\gamma,\alpha} \circ \rho_{\delta,\gamma} \ .$$

Proof. First we show that $\rho_{\delta,\gamma}$ is well-defined. To this end, assume that $[x]_\delta = [y]_\delta$. Then there exist $t \in 1 + \mathcal{M}^\delta$ such that $x = yt$. Since $\gamma \leq \delta$ we have that $1 + \mathcal{M}^\delta \subseteq 1 + \mathcal{M}^\gamma$, so we obtain that $x = yt$ for some $t \in 1 + \mathcal{M}^\gamma$ and thus

$$\rho_{\delta,\gamma}([x]_\delta) = [x]_\gamma = [y]_\gamma = \rho_{\delta,\gamma}([y]_\delta).$$

It is clear that (HH1) holds and that $\rho_{\delta,\gamma}$ is surjective. Furthermore, we have that

$$\rho_{\delta,\gamma}([x]_\delta[y]_\delta^{-1}) = \rho_{\delta,\gamma}([xy^{-1}]_\delta) = [xy^{-1}]_\gamma = [x]_\gamma[y]_\gamma^{-1} = \rho_{\delta,\gamma}([x]_\delta)\rho_{\delta,\gamma}([y]_\delta)^{-1}$$

for all $x, y \in K$ with $y \neq 0$.

In order to show (HH3), we compute

$$\begin{aligned} \rho_{\delta,\gamma}([x]_\delta + [y]_\delta) &= \{\rho_{\delta,\gamma}([x + yt]_\delta) \mid t \in 1 + \mathcal{M}^\delta\} \\ &= \{[x + yt]_\gamma \mid t \in 1 + \mathcal{M}^\delta\} \\ &\subseteq \{[x + yt]_\gamma \mid t \in 1 + \mathcal{M}^\gamma\} \\ &= [x]_\gamma + [y]_\gamma \\ &= \rho_{\delta,\gamma}([x]_\delta) + \rho_{\delta,\gamma}([y]_\delta). \end{aligned}$$

where we used again the fact that $1 + \mathcal{M}^\delta \subseteq 1 + \mathcal{M}^\gamma$. This proves that $\rho_{\delta,\gamma}$ is a homomorphism of hyperfields.

To show that $\rho_{\delta,\gamma}$ is value-preserving simply note that for all $[x]_\delta \in \mathcal{H}_\delta(K)$

$$v_\delta[x]_\delta = vx = v_\gamma[x]_\gamma = v_\gamma(\rho_{\delta,\gamma}[x]_\delta).$$

The last assertion of the lemma is clear. □

Remark 3.35. We note that $\ker \rho_{\delta,\gamma} = \{[0]_\delta\}$ for all $\gamma, \delta \in vK$ such that $0 \leq \gamma \leq \delta$. However, $\rho_{\delta,\gamma}$ might be not injective. An example is given by taking $K = \mathbb{Q}$ with v being the 2-adic valuation, $\delta = 1$ and $\gamma = 0$. It suffices to consider $[6]_1 \neq [2]_1$ and observe that $[6]_0 = [2]_0$, since 3 is not a 1-unit of level 1.

It is known that there are homomorphisms of hyperrings with trivial kernel which are not injective, for other examples the reader can see [53].

A valued field (K, v) admits a system of *projections* onto its γ-valued hyperfields, for $\gamma \in vK_{\geq 0}$:

$$\begin{aligned} p_\gamma : K &\to \mathcal{H}_\gamma(K) \\ x &\mapsto [x]_\gamma \end{aligned}$$

These projections are homomorphisms of hyperfields such that $vx = v_\gamma[x]_\gamma$ for all $x \in K$. Moreover, they respect the maps $\rho_{\delta,\gamma}$ in the sense that, for all non-negative $\gamma, \delta \in vK$ we have that

$$\begin{array}{ccc} & K & \\ {}^{p_\delta}\swarrow & & \searrow^{p_\gamma} \\ \mathcal{H}_\delta(K) & \xrightarrow[\rho_{\delta,\gamma}]{} & \mathcal{H}_\gamma(K) \end{array}$$

commutes.

We will now consider the completion (K^c, v) of a valued field (K, v). It is known (see e.g. [15, Theorem 2.4.3]) that (K, v) lies dense in (K^c, v) with respect to the ultrametric induced by the valuation. From this density, it follows that the value group vK^c and the residue field $K^c v$ coincide with vK and Kv, respectively. Moreover, we can make the following observation.

Lemma 3.36. *Let (K, v) be a valued field and (K^c, v) be its completion. Then for all $\gamma \in vK_{\geq 0}$, we have that $\mathcal{H}_\gamma(K) = \mathcal{H}_\gamma(K^c)$.*

Proof. Fix $\gamma \in vK_{\geq 0}$. By Part 1) of Lemma 3.3, for all nonzero $a \in K^c$, we have that $[a]_\gamma$, as a subset of K^c, is an open ultrametric ball. By the density of K in K^c, it follows that there exists $b \in K$ such that $b \in [a]_\gamma$ and the result follows. □

Thus, the valued field (K^c, v) has the same system of projections onto its γ-valued hyperfields as (K, v). We now notice that it is possible to characterize the extensions of (K, v) which embeds into its completion as those that have the same system of projections onto their γ-valued hyperfields as (K, v). This is the content of the next theorem.

Theorem 3.37. *Fix a valued field (K, v). Let (L, w) be an extension of (K, v) with $wL = vK$. Then (L, w) admits an embedding into (K^c, v) if and only if $\mathcal{H}_\gamma(L) = \mathcal{H}_\gamma(K)$ for all $\gamma \in vK_{\geq 0}$.*

Proof. Under the hypotheses of the theorem, let us first assume that (L, w) embeds into (K^c, v). We may identify L with its image under the given embedding. Since L contains K which lies dense in K^c, it follows that L lies dense in K^c. As in the proof of the previous lemma, we obtain that

$$\mathcal{H}_\gamma(L) = \mathcal{H}_\gamma(K^c) = \mathcal{H}_\gamma(K)$$

for all $\gamma \in vK_{\geq 0}$.

Conversely, let $(\gamma_\nu)_{\nu<\kappa}$ be an increasing and cofinal sequence of non-negative elements in vK. For any $a \in L^\times$ and all $\nu < \kappa$, we take $b_\nu \in K$ such that $[b_\nu]_{\gamma_\nu} = [a]_{\gamma_\nu}$. These elements exists since by assumption $\mathcal{H}_{\gamma_\nu}(L) = \mathcal{H}_{\gamma_\nu}(K)$ for all $\nu < \kappa$. Moreover, for all $\nu < \kappa$ we have that $vb_\nu = wa$. From the fact that $(\gamma_\nu)_{\nu<\kappa}$ is increasing and by Part 1) of Lemma 3.3, we obtain that

$$v(b_\nu - b_\mu) > \gamma_\nu + wa \quad \text{for all } \nu < \mu < \kappa.$$

Since $(\gamma_\nu)_{\nu<\kappa}$ is cofinal in vK, this implies that $(b_\nu)_{\nu<\kappa}$ is a Cauchy sequence in K and it then has a limit $b \in K^c$. We claim that b does not depend on the choice of $b_\nu \in K$. For let $c_\nu \in K$ such that $[c_\nu]_{\gamma_\nu} = [a]_{\gamma_\nu}$ be another choice. As above, $(c_\nu)_{\nu<\kappa}$ is a Cauchy sequence in K and we denote by c its limit in K^c. Fix $\delta \in vK$ and let $\nu < \kappa$ be such that $\gamma_\nu + wa > \delta$ and $v(b - b_\nu), v(c_\nu - c) > \delta$. By Part 1) of Lemma 3.3, since $vc_\nu = wa = vb_\nu$, we obtain that

$$v(b_\nu - c_\nu) > \gamma_\nu + wa > \delta$$

and then

$$v(b - c) = v(b - b_\nu + b_\nu - c_\nu + c_\nu - c) \geq \min\{v(b - b_\nu), v(b_\nu - c_\nu), v(c_\nu - c)\} > \delta.$$

Since δ is arbitrary, we conclude that $c = b$.

We now claim that the assignment $a \mapsto b$ defines an embedding of valued fields

$$\sigma : L \to K^c.$$

Let $a, b \in L$ and assume without loss of generality that $wa \leq wb$. Take $c_\nu, x_\nu, y_\nu \in K$ such that $[c_\nu]_{\gamma_\nu} = [a + b]_{\gamma_\nu}$, $[x_\nu]_{\gamma_\nu} = [a]_{\gamma_\nu}$ and $[y_\nu]_{\gamma_\nu} = [b]_{\gamma_\nu}$ for all $\nu < \kappa$. As before, these elements form Cauchy sequences in K and we denote by c, x and y their respective limits in K^c. Thus, $\sigma(a + b) = c$, $\sigma(a) = x$ and $\sigma(b) = y$. We now prove that $c = x + y$. We first observe that, by Part 4) of Lemma 3.3, for all $\nu < \kappa$,

$$c_\nu \in \bigcup([x_\nu]_{\gamma_\nu} + [y_\nu]_{\gamma_\nu}).$$

Now, fix $\delta \in vK$ and let $\nu < \kappa$ be large enough so that $\gamma_\nu + wa > \delta$ and $v(x - x_\nu), v(y - y_\nu), v(c - c_\nu) > \delta$. Applying Part 2) of Lemma 3.3 we obtain that

$$v(x_\nu + y_\nu - c_\nu) > \gamma_\nu + wa > \delta,$$

where we used the fact that $vx_\nu = wa$ for all $\nu < \kappa$. Then we obtain that

$$v(x_\nu + y_\nu - c) = v(x_\nu + y_\nu - c_\nu + c_\nu - c) \geq \min\{v(x_\nu + y_\nu - c_\nu), v(c_\nu - c)\} > \delta.$$

Therefore, we have that

$$v(x+y-c) = v(x-x_\nu+y-y_\nu+x_\nu+y_\nu-c) \geq \min\{v(x-x_\nu), v(y-y_\nu), v(x_\nu+y_\nu-c)\} > \delta.$$

Since δ is arbitrary, we conclude that $c = x + y$, as we wished to show.

For all $a, b \in L$ with $b \neq 0$, we have that $[-a]_\gamma = -[a]_\gamma$ and that $[ab^{-1}]_\gamma = [a]_\gamma [b]_\gamma^{-1}$ for all $\gamma \in vK_{\geq 0}$. From this, it easily follows that $\sigma(-a) = -\sigma(a)$ and that $\sigma(ab^{-1}) = \sigma(a)\sigma(b)^{-1}$.

Therefore, σ is a homomorphism of fields and is thus injective. It remains to prove that σ preserves the valuations. We claim that $wa = v\sigma(a)$ for all $a \in L$. This is because, as $[b_\nu]_{\gamma_\nu} = [a]_{\gamma_\nu}$, we have that $vb_\nu = wa$ for all $\nu < \kappa$. Therefore, the limit $b = \sigma(a)$ of $(b_\nu)_{\nu<\kappa}$ must have the same value. □

We observe that in the above proof, when showing that (L, w) can be embedded into the complete valued field (K^c, v), we have not used the assumption that (L, w) is an extension of (K, v). This yields the following corollary.

Corollary 3.38. *Fix a complete valued field (K, v). If (L, w) is any valued field with $wL = vK$ and $\mathcal{H}_\gamma(L) = \mathcal{H}_\gamma(K)$ for all $\gamma \in vK_{\geq 0}$, then (L, w) embeds into (K, v).*

Fix an ordered abelian group Γ. We define the category Γ-**Vhyp** to have as objects valued hyperfields with value group Γ. As a *morphism* between two objects (F, u) and (F', u') we take a homomorphism of hyperfields $\rho : F \to F'$ such that $ux = u'\rho(x)$ for all $x \in F$.

The projective system given by Lemma 3.34 can be understood as a *diagram* in vK-**Vhyp**. The valued field (K, v), together with all of the projections onto its γ-valued hyperfields, forms a *cone* on this diagram in vK-**Vhyp**. For a reference on basic category theory the reader can see [33, Section 5.1] from which we took the terminology used here.

Lemma 3.39. *Fix a valued field (K, v) and consider the diagram given by all of its γ-valued hyperfields in the category vK-**Vhyp**. Let (L, w) be any cone on this diagram. Then L is a field, i.e., for all $x, y \in L$ we have that $x + y$ is a singleton.*

Proof. For all $\gamma \in vK_{\geq 0}$, we denote by $f_\gamma : L \to \mathcal{H}_\gamma(K)$ the morphism of vK-**Vhyp** associated to the given cone. Pick $x, y \in L^\times$ and let $z, z' \in x + y$. We claim that $0_L \in z - z'$, implying that $z = z'$ and thus that $x + y$ is a singleton. Since f_γ satisfies (HH3), for all $\gamma \in vK_{\geq 0}$ we have that $f_\gamma(z), f_\gamma(z') \in f_\gamma(x) + f_\gamma(y)$

holds in $\mathcal{H}_\gamma(K)$. Since $wx = v_\gamma f_\gamma(x)$ and $wy = v_\gamma f_\gamma(y)$ for all $\gamma \in vK_{\geq 0}$, applying Proposition 3.26 we conclude that $f_\gamma(x) + f_\gamma(y)$ is an open ultrametric ball of radius $\gamma + \min\{wx, wy\}$, for all $\gamma \in vK_{\geq 0}$. Let $(\gamma_\nu)_{\nu<\kappa}$ be an increasing and cofinal sequence of non-negative elements of vK and fix $\delta \in vK$. Let ν be large enough so that $\gamma_\nu + \min\{wx, wy\} > \delta$. Now, suppose that $[0]_{\gamma_\nu} \notin f_{\gamma_\nu}(z) - f_{\gamma_\nu}(z')$. By Proposition 3.19 and since f_{γ_ν} satisfies (HH3), we conclude that for any $a \in z - z'$, the value $wa = v_{\gamma_\nu} f_{\gamma_\nu}(a)$ is larger than δ. Since δ is arbitrary, this implies that $a = 0_L$; but then

$$[0]_{\gamma_\nu} = f_{\gamma_\nu}(a) \in f_{\gamma_\nu}(z) - f_{\gamma_\nu}(z').$$

This contradiction shows that $[0]_{\gamma_\nu} \in f_{\gamma_\nu}(z) - f_{\gamma_\nu}(z')$ must hold and therefore $f_{\gamma_\nu}(z) = f_{\gamma_\nu}(z')$. In particular, $wz = wz'$ and by enlarging ν if necessary we can ensure that $\gamma_\nu + wz > \delta$ as well. Now, for all $a \in z - z'$ we will have that $f_{\gamma_\nu}(a) \in f_{\gamma_\nu}(z) - f_{\gamma_\nu}(z)$ and by Proposition 3.26, $f_{\gamma_\nu}(z) - f_{\gamma_\nu}(z)$ is an open ultrametric ball of radius $\gamma_\nu + wz$ and center $[0]_{\gamma_\nu}$. Hence, $wa = v_{\gamma_\nu} f_{\gamma_\nu}(a)$ will be larger than δ and since δ is arbitrary, $a = 0_L$ follows. This completes the proof. □

Fix a valued field (K, v) and let (L, w) be any cone on the diagram of the γ-valued hyperfields of (K, v) in vK-**Vhyp**. By the previous lemma (L, w) is a valued field and by Corollary 3.38, it embeds into the completion (K^c, v) of (K, v).

Let us denote by $f_\gamma : L \to \mathcal{H}_\gamma(K)$ and by $\tilde{p}_\gamma : K^c \to \mathcal{H}_\gamma(K)$ the projections onto the γ-valued hyperfields of K of (L, w) and (K^c, v), respectively.

It is straightforward to verify that the embedding σ that we have constructed in the proof of Theorem 3.37 is unique with the property that $f_\gamma = \tilde{p}_\gamma \circ \sigma$ for all $\gamma \in vK_{\geq 0}$. This follows from the fact that for any $a \in L^\times$, the classes $([a]_\gamma)_{\gamma \in vK_{\geq 0}}$ form a chain of balls of increasing radii and that the set of this radii is cofinal in vK. Moreover, σ is a morphism in vK-**Vhyp**. We deduce the following result.

Theorem 3.40. *For any valued field (K, v), its completion (K^c, v) is the limit cone in vK-**Vhyp** on the diagram*

$$\left(\mathcal{H}_\delta(K) \xrightarrow{\rho_{\delta,\gamma}} \mathcal{H}_\gamma(K)\right)_{\delta \geq \gamma \geq 0} .$$

Chapter 4

The γ-valued hyperfields and other structures

We are interested in the model theory of valued fields, in particular in relative quantifier elimination results. Such results have been obtained for some classes of valued fields relatively to several different structures. In this chapter we focus our attention to the relations between these structures and the γ-valued hyperfields.

In the first section we consider the leading term structures which Flenner studies in [18] (see also [17]). We note that they are basically the same thing as the γ-valued hyperfields. The difference is a matter of description of the hyperoperation which in the leading term structures is encoded via a ternary relation. Since from the model theoretical point of view we indeed describe the hyperoperation with a ternary relation symbol, the two structures result equivalent for our purposes.

In the second section we take into account the amc-structures of Kuhlmann (see [26]). In [18], Flenner showed that there is a biinterpretability relation between these structures and his leading term structures. We take here a more algebraic approach and using our observations from the previous section and some ideas of Flenner, we develop a tight relation between the amc-structure of level γ and the γ-valued hyperfield of any valued field.

These sections contain joint work with P. Błaszkiewicz.

In the third section we turn our attention to angular component maps which are also a tool used to obtain relative quantifier elimination results for valued fields. We study what happens to the structure of $\mathcal{H}_0(K)$ when K admits an angular component map.

In the last section we take into account the graded ring of a valued field. This structure is used in algebraic geometry more than in model theory (see for instance [40]). We relate this structure with the structure of the 0-valued hyperfield.

4.1 Leading term structures

Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. The *leading term structure of level* γ or simply the *RV_γ-structure* is defined in [17] by Flenner as

$$RV_\gamma := K^\times/(1 + \mathcal{M}^\gamma) \cup \{\mathbf{0}\}$$

Note that this set coincides with $\mathcal{H}_\gamma(K)$ once we identify $\mathbf{0}$ with $[0]_\gamma$. On RV_γ a ternary relation is introduced:

$$\oplus_\gamma(\mathbf{x}, \mathbf{y}, \mathbf{z}) \iff \exists x, y, z \in K : rv_\gamma(x) = \mathbf{x} \wedge rv_\gamma(y) = \mathbf{y} \wedge rv_\gamma(z) = \mathbf{z} \wedge x + y = z$$

where $rv_\gamma : K \to RV_\gamma$ is the canonical projection on $K^\times$ and sends 0 to $\mathbf{0}$. Note that for $x \in K^\times$ we have that $rv_\gamma(x) = [x]_\gamma$.

The following result connects the ternary relation introduced above with the hyperoperation of $\mathcal{H}_\gamma(K)$.

Theorem 4.1. *Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. Take $a, b, c \in K$. Then $\oplus_\gamma(rv_\gamma(a), rv_\gamma(b), rv_\gamma(c))$ holds in RV_γ if and only if $[c]_\gamma \in [a]_\gamma + [b]_\gamma$ holds in $\mathcal{H}_\gamma(K)$.*

Proof. Note that

$$[a]_\gamma + [b]_\gamma = \{[x + y]_\gamma \in \mathcal{H}_\gamma(K) \mid x \in [a]_\gamma \ , y \in [b]_\gamma\}.$$

This follows easily from the definitions and Lemma 2.9. On the other hand, by definition we have that $\oplus_\gamma(rv_\gamma(a), rv_\gamma(b), rv_\gamma(c))$ holds in RV_γ if and only if there are $x, y, z \in K$ such that $rv_\gamma(x) = rv_\gamma(a)$, $rv_\gamma(y) = rv_\gamma(b)$, $rv_\gamma(z) = rv_\gamma(c)$ and $x + y = z$. Notice that since $rv_\gamma(a)$, $rv_\gamma(b)$ and $rv_\gamma(c)$ are equivalence classes, this means that there are $x, y, z \in K$ such that $x \in rv_\gamma(a)$, $y \in rv_\gamma(b)$, $z \in rv_\gamma(c)$ and $x + y = z$. We distinguish several cases.

In the first case we assume that $a, b, c \in K^\times$. Hence, $rv_\gamma(a) = [a]_\gamma$, $rv_\gamma(b) = [b]_\gamma$ and $rv_\gamma(c) = [c]_\gamma$. We then have that $[c]_\gamma \in [a]_\gamma + [b]_\gamma$ holds in $\mathcal{H}_\gamma(K)$ if and only if $x + y = z$ for some $x \in [a]_\gamma = rv_\gamma(a)$, $y \in [b]_\gamma = rv_\gamma(b)$ and $z \in [c]_\gamma = rv_\gamma(c)$. This means that $\oplus_\gamma(rv_\gamma(a), rv_\gamma(b), rv_\gamma(c))$ holds in RV_γ.

In the second case we assume that $a = 0$ and that $b, c \in K^\times$. In this case, $[c]_\gamma \in [a]_\gamma + [b]_\gamma$ is equivalent to $[c]_\gamma = [b]_\gamma$. We have to show that this is equivalent to $\oplus_\gamma(\mathbf{0}, rv_\gamma(b), rv_\gamma(c))$. This is done as follows: since $rv_\gamma(x) = \mathbf{0}$ if and only if $x = 0$, we obtain that $\oplus_\gamma(\mathbf{0}, rv_\gamma(b), rv_\gamma(c))$ holds in RV_γ if and only if there are $y, z \in K$ such that $y \in rv_\gamma(b) = [b]_\gamma$, $z \in rv_\gamma(c) = [c]_\gamma$ and $0 + y = z$, i.e., $z - y = 0$. This means that $[0]_\gamma \in [c]_\gamma - [b]_\gamma$ or, in other words, $[b]_\gamma = [c]_\gamma$.

The third case, where $b = 0$ and $a, c \in K^\times$ is analogous.

In the fourth case, we assume that $c = 0$ and that $a, b \in K^\times$. Now we have to show that $[0]_\gamma \in [a]_\gamma + [b]_\gamma$ is equivalent to $\oplus_\gamma(rv_\gamma(a), rv_\gamma(b), \mathbf{0})$. Again since

$rv_\gamma(z) = 0$ if and only if $z = 0$, we obtain that $\oplus_\gamma(rv_\gamma(a), rv_\gamma(b), \mathbf{0})$ holds in RV_γ if and only if there are $x, y \in K$ such that $x \in rv_\gamma(a) = [a]_\gamma$, $y \in rv_\gamma(b) = [b]_\gamma$ and $x + y = 0$. This means that $[0]_\gamma \in [a]_\gamma + [b]_\gamma$.

We now analyze the remaining cases: if two elements between a, b and c are 0 and the third is nonzero in K, then the two assertions which we claim are equivalent are both false. If $a = b = c = 0$, then the assertions which we claim to be equivalent are both true. Now the proof is complete. □

The *language* $\mathcal{L}_{RV}$ *of RV-structures* is the language of (multiplicative) groups extended with a ternary relation symbol r_+ to be interpreted as $\oplus_\gamma$. As we already noted, the language of valued hyperfields is an extension of this language with the unary function symbol $-$ and the unary relation symbol $\mathcal{O}$. The previous result shows that interpreting $r_+(x, y, z)$ as $z \in x + y$ in $\mathcal{H}_\gamma(K)$ is the same thing as interpreting it as $\oplus_\gamma(x, y, z)$ in RV_γ. Therefore, if (K, v) is a valued field and $\gamma \in vK_{\geq 0}$, then RV_γ and $\mathcal{H}_\gamma(K)$ are the same thing as $\mathcal{L}_{RV}$-structures.

4.2 amc-structures

In [26] Kuhlmann introduces the amc-structures of level γ for a valued field (K, v) and $\gamma \in vK_{\geq 0}$. They consist of the residue ring $\mathcal{O}^\gamma := \mathcal{O}_v/\mathcal{M}^\gamma$ and the group $G^\gamma := K^\times/(1+\mathcal{M}^\gamma)$ plus a relation Θ_γ between them which we now define. Write π_γ for the canonical projection map $\mathcal{O}_v \to \mathcal{O}^\gamma$ and π_γ^* for the canonical projection map $K^\times \to G^\gamma$. Then

$$\Theta_\gamma := \{(x, y) \in \mathcal{O}^\gamma \times G^\gamma \mid \exists z \in \mathcal{O}_v : \pi_\gamma z = x \wedge \pi_\gamma^* z = y\}.$$

We denote the amc-structure $(\mathcal{O}^\gamma, G^\gamma, \Theta_\gamma)$ of level γ of (K, v) by K_γ.

We will sometimes extend the relation Θ_γ to $G^\gamma \cup \{0\} = \mathcal{H}_\gamma(K)$, by setting

$$\Theta_\gamma(x, 0) \iff x = 0$$

for all $x \in \mathcal{O}^\gamma$.

Since G^γ is a reduct of $\mathcal{H}_\gamma(K)$, it is clear that all the information contained in G^γ can be recovered from the γ-valued hyperfield of (K, v). For the residue ring $\mathcal{O}^\gamma$ we can apply Proposition 3.15 with $\delta = \gamma$ to obtain the following result.

Proposition 4.2. *Let* (K, v) *be a valued field and* $\gamma \in vK_{\geq 0}$ *. Then* $\mathcal{H}_\gamma(\mathcal{M}^\gamma)$ *is a hyperideal of* $\mathcal{H}_\gamma(\mathcal{O}_v)$ *and the* γ*-residue ring* $\mathcal{O}^\gamma$ *of* (K, v) *is isomorphic to the quotient hyperring* $\mathcal{H}_\gamma(\mathcal{O}_v)/\mathcal{H}_\gamma(\mathcal{M}^\gamma)$*.*

However, the relation between the amc-structures and the γ-valued hyperfields is even tighter. Before pushing this forward we introduce the following notion of isomorphism of valued hyperfields.

Definition 4.3. An *isomorphism of valued hyperfields* from (F, v) onto (F', v') is an isomorphism $\sigma : F \to F'$ of hyperfields such that $\sigma\mathcal{O}_v = \mathcal{O}_{v'}$.

Remark 4.4. Note that this notion is consistent with Definition 1.6 when we consider valued hyperfields as $\mathcal{L}_{vh}$-structures.

Fix valued fields (K, v) and (L, w). Take $\gamma \in vK_{\geq 0}$ and $\delta \in wL_{\geq 0}$. Consider the amc-structures K_γ and L_δ and assume that they are isomorphic. Does it follow that $(\mathcal{H}_\gamma(K), v_\gamma) \simeq (\mathcal{H}_\delta(L), w_\delta)$ as valued hyperfields? What about the converse? To provide an answer to both of these questions we need some preparations.

Lemma 4.5. *Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. Then*

$$[1]_\gamma - [1]_\gamma = \mathcal{H}_\gamma(\mathcal{M}^\gamma).$$

Proof. By Part 2) of Lemma 3.3 we have that $[x]_\gamma \in [1]_\gamma - [1]_\gamma$ if and only if

$$vx = v(x + (1 - 1)) > \gamma + \min\{v1, v(-1)\} = \gamma.$$

This proves the claim. □

Corollary 4.6. *Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. For all $[x]_\gamma \in \mathcal{H}_\gamma(K)$ we have that*

$$v_\gamma[x]_\gamma > \gamma \iff [1]_\gamma \in [x]_\gamma + [1]_\gamma$$

Proof. We have that $vx = v_\gamma[x]_\gamma > \gamma$ if and only if $[x]_\gamma \in \mathcal{H}_\gamma(\mathcal{M}^\gamma)$ and by the previous lemma this happens if and only if $[x]_\gamma \in [1]_\gamma - [1]_\gamma$. By the reversibility axiom (H4), the latter is equivalent to $[1]_\gamma \in [x]_\gamma + [1]_\gamma$. □

We observe that $\pi^*_\gamma x = [x]_\gamma$ for all $x \in K^\times$. Therefore, we will switch from one notation to the other whenever convenient.

Lemma 4.7. *Let (K, v) be a valued field and fix $\gamma \in vK_{\geq 0}$. For $x \in \mathcal{O}_v$ and $y \in K^\times$ we have that*

$$K_\gamma \models \Theta_\gamma(x + \mathcal{M}^\gamma, [y]_\gamma) \iff \mathcal{H}_\gamma(K) \models [x]_\gamma \sim_I [y]_\gamma.$$

where $I := \mathcal{H}_\gamma(\mathcal{M}^\gamma)$.

Proof. If there exists $z \in \mathcal{O}_v$ such that $x + \mathcal{M}^\gamma = z + \mathcal{M}^\gamma$ and $[z]_\gamma = [y]_\gamma$, then $[x - z]_\gamma \in ([x]_\gamma - [y]_\gamma) \cap I$ and thus $[x]_\gamma \sim_I [y]_\gamma$.

For the converse, assume that $[x]_\gamma \sim_I [y]_\gamma$. We first note that $y \in \mathcal{O}_v$ must hold. Indeed, suppose not, then for all $[z]_\gamma \in [x]_\gamma - [y]_\gamma$ we would have that

$$v_\gamma[z]_\gamma = \min\{v_\gamma[x]_\gamma, v_\gamma[y]_\gamma\} = v_\gamma[y]_\gamma < 0$$

and then $[x]_\gamma \sim_I [y]_\gamma$ would not hold since $\gamma \geq 0$. Now let $[xt - yu]_\gamma \in [x]_\gamma - [y]_\gamma$ (cf. Lemma 2.9) be such that $v(xt - yu) > \gamma$. Write $t = 1 + c$ and $u = 1 + d$ for some $c, d \in \mathcal{M}^\gamma$. We obtain, $xt-yu = x-y+xc-yd$, thus $x-y = xt-yu+yd-xc$ and then

$$v(x - y) \geq \min\{v(xt - yu), v(yd - xc)\} > \gamma$$

since $v(yd - xc) \geq \min\{vy + vd, vx + vc\} > \gamma$. Therefore, $x + \mathcal{M}^\gamma = y + \mathcal{M}^\gamma$. We have proved that $\Theta_\gamma(x + \mathcal{M}^\gamma, [y]_\gamma)$ holds in K_γ (take $z = y \in \mathcal{O}_v$ in the definition of Θ_γ). □

Lemma 4.8. *Pick $a \in K^\times$. For all $b \in K$ such that $va \leq vb$ we have that $[c]_\gamma \in [a]_\gamma + [b]_\gamma$ holds in $\mathcal{H}_\gamma(K)$ if and only if there exist $\bar{x}, \bar{y} \in \mathcal{O}^\gamma$ such that*

$$\Theta_\gamma(\bar{x}, [ba^{-1}]_\gamma) \wedge \Theta_\gamma(\bar{y}, [ca^{-1}]_\gamma) \wedge \bar{1} + \bar{x} = \bar{y} \tag{4.1}$$

holds in K_γ.

Proof. Take $a \in K^\times$ and $b \in K$ with $va \leq vb$. If $b = 0$, then $[c]_\gamma \in [a]_\gamma + [b]_\gamma$ means that $[c]_\gamma = [a]_\gamma$ and we may take $\bar{x} = \bar{0}$ and $\bar{y} = \bar{1}$.

Hence, we consider $a, b \in K^\times$ with $va \leq vb$. If $[c]_\gamma \in [a]_\gamma + [b]_\gamma$, then multiplying by $[a]_\gamma^{-1} = [a^{-1}]_\gamma$ and using (R3) we obtain $[ca^{-1}]_\gamma \in [1]_\gamma + [ba^{-1}]_\gamma$. By our assumption and by (V3), we have that $ca^{-1}, ba^{-1} \in \mathcal{O}_v$. Since $1 \in \mathcal{O}_v$, we can apply Part 5) of Lemma 3.3 to obtain that $ca^{-1} = 1 + ba^{-1} + d$ for some $d \in \mathcal{M}^\gamma$. Therefore,

$$\pi_\gamma(1) + \pi_\gamma(ba^{-1}) = \pi_\gamma(1 + ba^{-1}) = \pi_\gamma(ca^{-1})$$

which means that $\bar{y} := \pi_\gamma(ca^{-1})$ and $\bar{x} := \pi_\gamma(ba^{-1})$ satisfy (4.1). Indeed, notice that $\Theta_\gamma(\bar{x}, [ba^{-1}]_\gamma)$ holds since $ba^{-1} \in \mathcal{O}_v$ and $\Theta_\gamma(\bar{y}, [ca^{-1}]_\gamma)$ holds since $ca^{-1} \in \mathcal{O}_v$.

For the other implication assume that there exist $\bar{x}, \bar{y} \in \mathcal{O}^\gamma$ such that

$$\Theta_\gamma(\bar{x}, [ba^{-1}]_\gamma) \wedge \Theta_\gamma(\bar{y}, [ca^{-1}]_\gamma) \wedge \bar{1} + \bar{x} = \bar{y}$$

holds in K_γ for $a, b, c \in K$ with $a \neq 0$ and $va \leq vb$. Since $\Theta_\gamma(\bar{y}, [ca^{-1}]_\gamma)$ holds, we find $y \in \mathcal{O}_v$ such that $\pi_\gamma y = \bar{y}$ and $[y]_\gamma = [ca^{-1}]$. Similarly, since since $\Theta_\gamma(\bar{x}, [ba^{-1}]_\gamma)$ holds, we find $x \in \mathcal{O}_v$ such that $\pi_\gamma x = \bar{x}$ and $[x]_\gamma = [ba^{-1}]_\gamma$. Now we have that $\bar{1} + \bar{x} = \bar{y}$ which implies $\pi_\gamma(1 + x) = \pi_\gamma y$. Therefore,

$$v(y - 1 - x) > \gamma = \gamma + \min\{vx, v1\}.$$

By Part 2) of Lemma 3.3 we conclude that

$$[y]_\gamma \in [1]_\gamma + [x]_\gamma \ .$$

Multiplying by $[a]_\gamma$ and using (R3), we obtain

$$[y]_\gamma[a]_\gamma \in [a]_\gamma + [x]_\gamma[a]_\gamma \ .$$

Finally, since $[y]_\gamma = [ca^{-1}]_\gamma$ and $[x]_\gamma = [ba^{-1}]_\gamma$ we find that $[c]_\gamma \in [a]_\gamma + [b]_\gamma$ as we wished to show. □

Assume now that $K_\gamma \simeq L_\delta$. An *isomorphism* σ *of amc-structures* consists of two maps $\sigma_r : \mathcal{O}_K^\gamma \to \mathcal{O}_L^\delta$ and $\sigma_g : G_K^\gamma \to G_L^\delta$ which are isomorphisms in the language of rings and of groups, respectively and such that for all $x \in \mathcal{O}_K^\gamma$ and $y \in G_K^\gamma$

$$\Theta_\gamma(x, y) \iff \Theta_\delta(\sigma_r x, \sigma_g y). \tag{4.2}$$

Note that then $(\sigma_r^{-1}, \sigma_g^{-1})$ is also an isomorphism of amc-structures which is the *inverse* of (σ_r, σ_g). We will now show that σ_g extended to $\mathcal{H}_\gamma(K)$ so that $\sigma_g(0) = 0$ is an isomorphism of valued hyperfields. Note that this extension is still a bijective map.

We first show that

$$\sigma_g \mathcal{H}_\gamma(\mathcal{O}_v) = \mathcal{H}_\delta(\mathcal{O}_w) \tag{4.3}$$

Take $[y]_\gamma \in \mathcal{H}_\gamma(\mathcal{O}_v)$. If $[y]_\gamma = [0]_\gamma$, then $\sigma_g[y]_\gamma = [0]_\delta \in \mathcal{H}_\delta(\mathcal{O}_w)$. Otherwise, we have $y \in \mathcal{O}_v \setminus \{0\}$ and $\Theta_\gamma(y + \mathcal{M}^\gamma, [y]_\gamma)$ holds. Therefore, $\Theta_\delta(\sigma_r(y + \mathcal{M}^\gamma), \sigma_g[y]_\gamma)$ holds, but this implies that $\sigma_g[y]_\gamma \in \mathcal{H}_\delta(\mathcal{O}_w)$ by the definition of Θ_δ. This shows $\sigma_g \mathcal{H}_\gamma(\mathcal{O}_v) \subseteq \mathcal{H}_\delta(\mathcal{O}_w)$. Applying the same reasoning to σ_g^{-1} we obtain that

$$\sigma_g^{-1} \mathcal{H}_\delta(\mathcal{O}_w) \subseteq \mathcal{H}_\gamma(\mathcal{O}_v).$$

Thus,

$$\mathcal{H}_\delta(\mathcal{O}_w) \subseteq \sigma_g \mathcal{H}_\gamma(\mathcal{O}_v)$$

follows by an application of σ_g.

Now we show that for all $[x]_\gamma, [y]_\gamma \in \mathcal{H}_\gamma(K)$ we have that

$$\sigma_g([x]_\gamma + [y]_\gamma) = \sigma_g[x]_\gamma + \sigma_g[y]_\gamma. \tag{4.4}$$

If $x = 0$ or $y = 0$, then the claim is obvious. Hence, we may assume that $x, y \neq 0$ and without loss of generality that $vx \leq vy$. Take $[z]_\delta \in \sigma_g([x]_\gamma + [y]_\gamma)$. This is equivalent to $\sigma_g^{-1}[z]_\delta \in [x]_\gamma + [y]_\gamma$. By Lemma 4.8 the latter happens if and only if

$$\exists \bar{a}, \bar{b} \in \mathcal{O}_K^\gamma \ : \ \Theta_\gamma(\bar{a}, [x^{-1}y]_\gamma) \wedge \Theta_\gamma(\bar{b}, [x^{-1}]_\gamma \sigma_g^{-1}[z]_\delta) \wedge \bar{1} + \bar{a} = \bar{b}$$

holds in K_γ. This in turn holds if and only if

$$\exists \bar{a}, \bar{b} \in \mathcal{O}_K^\gamma \ : \ \Theta_\delta(\sigma_r \bar{a}, \sigma_g([x^{-1}y]_\gamma)) \wedge \Theta_\delta(\sigma_r \bar{b}, \sigma_g([x^{-1}]_\gamma)[z]_\delta) \wedge \sigma_r(\bar{1} + \bar{a}) = \sigma_r \bar{b}$$

holds, where we used (4.2). However, this just means that

$$\exists \bar{a}', \bar{b}' \in \mathcal{O}_L^\delta \ : \ \Theta_\delta(\bar{a}', \sigma_g([x]_\gamma)^{-1}\sigma_g[y]_\gamma) \wedge \Theta_\delta(\bar{b}', \sigma_g([x]_\gamma)^{-1}[z]_\delta) \wedge \bar{1} + \bar{a}' = \bar{b}'$$

holds in L_δ. By Lemma 4.8 this is equivalent to $[z]_\delta \in \sigma_g[x]_\gamma + \sigma_g[y]_\gamma$. Note that $w_\delta \sigma_g[x]_\gamma \leq w_\delta \sigma_g[y]_\gamma$ since, as we have already shown, $\sigma_g \mathcal{O}_{v_\gamma} = \mathcal{O}_{w_\delta}$.

We have proved the following result.

Theorem 4.9. *Let (K, v) and (L, w) be valued fields and take $\gamma \in vK_{\geq 0}$ and $\delta \in wL_{\geq 0}$. If $K_\gamma \simeq L_\delta$, then $(\mathcal{H}_\gamma(K), v_\gamma) \simeq (\mathcal{H}_\delta(L), w_\delta)$.*

We are now interested in the converse. We then assume that σ is an isomorphism of valued hyperfields from $(\mathcal{H}_\gamma(K), v_\gamma)$ onto $(\mathcal{H}_\delta(L), w_\delta)$. Restricting σ to the nonzero elements of $\mathcal{H}_\gamma(K)$ we obtain an isomorphism (of groups) $\sigma_g : G_K^\gamma \to G_L^\delta$.

We now wish to construct an isomorphism of rings $\sigma_r : \mathcal{O}_K^\gamma \to \mathcal{O}_L^\delta$. For this, by Proposition 3.15, it suffices to induce from σ an isomorphism $R^\gamma/I^\gamma \to R^\delta/I^\delta$, where R^γ denotes the valuation hyperring of $\mathcal{H}_\gamma(K)$, I^γ its hyperideal $\mathcal{H}_\gamma(\mathcal{M}^\gamma)$, R^δ the valuation hyperring of $\mathcal{H}_\delta(L)$ and I^δ its hyperideal $\mathcal{H}_\delta(\mathcal{M}^\delta)$ (cf. also Proposition 4.2). Since σ is an isomorphism of valued hyperfields, we have $\sigma R^\gamma = R^\delta$. We define

$$\begin{aligned} \bar{\sigma} : R^\gamma/I^\gamma &\to R^\delta/I^\delta \\ [x]_{I^\gamma} &\mapsto [\sigma x]_{I^\delta} \end{aligned}$$

and we claim that it is an isomorphism of hyperrings.

First let us show that it is well-defined. Assume that $[x]_{I^\gamma} = [y]_{I^\gamma}$. Then $(x - y) \cap I^\gamma \neq \emptyset$ and we may pick $z \in (x - y) \cap I^\gamma$. By Corollary 4.6 we obtain that $1 \in z + 1$ so that

$$1 = \sigma(1) \in \sigma(z + 1) = \sigma z + \sigma(1) = \sigma z + 1.$$

This by Corollary 4.6 shows that $w_\delta \sigma z > \delta$ and thus $\sigma z \in I^\delta$. On the other hand, since $z \in x - y$ we have that $\sigma z \in \sigma x - \sigma y$. This shows that $(\sigma x - \sigma y) \cap I^\delta \neq \emptyset$ and therefore $[\sigma x]_{I^\delta} = [\sigma y]_{I^\delta}$ as required.

Now, let us show injectivity. For this, assume that $[\sigma x]_{I^\delta} = [\sigma y]_{I^\delta}$, then there exists $z' \in \sigma x - \sigma y$ such that $1 \in z' + 1$ by Corollary 4.6. Since $\sigma x - \sigma y = \sigma(x - y)$ we find $z \in x - y$ such that $\sigma z = z'$ and $\sigma(1) \in \sigma(z + 1)$ now yields that $1 \in z + 1$, since σ is bijective by assumption. Thus, $[x]_{I^\gamma} = [y]_{I^\gamma}$ by Corollary 4.6 as we wished to show.

Surjectivity is clear: it follows from the surjectivity of σ.

For the hyperaddition recall that, since σ is an isomorphism of hyperfields we have that $z \in x + y$ if and only if $\sigma z \in \sigma(x + y) = \sigma x + \sigma y$, for all $x, y, z \in \mathcal{H}_\gamma(K)$. Therefore, we have that

$$\begin{aligned} \bar{\sigma}([x]_{I^\gamma} + [y]_{I^\gamma}) &= \{\bar{\sigma}[z]_{I^\gamma} \mid z \in x + y\} \\ &= \{[\sigma z]_{I^\delta} \mid z \in x + y\} \\ &= \{[\sigma z]_{I^\delta} \mid \sigma z \in \sigma(x + y)\} \\ &= \{[z']_{I^\delta} \mid z' \in \sigma x + \sigma y\} \\ &= [\sigma x]_{I^\delta} + [\sigma y]_{I^\delta} \\ &= \bar{\sigma}[x]_{I^\gamma} + \bar{\sigma}[y]_{I^\gamma}. \end{aligned}$$

Finally, for the multiplication we have

$$\bar{\sigma}([x]_{I^\gamma}[y]_{I^\gamma}) = \bar{\sigma}[xy]_{I^\gamma} = [\sigma(xy)]_{I^\delta} = [\sigma x]_{I^\delta}[\sigma y]_{I^\delta} = \bar{\sigma}[x]_{I^\gamma}\bar{\sigma}[y]_{I^\gamma}.$$

Therefore, we also have an isomorphism of rings $\sigma_r : \mathcal{O}_K^\gamma \to \mathcal{O}_L^\delta$, by Proposition 4.2 as mentioned above.

It remains to show that for all $x \in \mathcal{O}_K^\gamma$ and $y \in G_K^\gamma$ we have that

$$\Theta_\gamma(x, y) \iff \Theta_\delta(\sigma_r x, \sigma_g y).$$

Let $a \in \mathcal{O}_v$ be such that $x = a + \mathcal{M}^\gamma$ and $b \in K^\times$ such that $y = [b]_\gamma$. Then, using Lemma 4.7 we obtain that $\Theta_\gamma(x, y)$ holds if and only if $[a]_\gamma \sim_{I^\gamma} [b]_\gamma$. This is equivalent to $\bar{\sigma}[a]_{\gamma,I^\gamma} = \bar{\sigma}[b]_{\gamma,I^\gamma}$ by the bijectivity of $\bar{\sigma}$. However, using the definition of $\bar{\sigma}$, we have that this means that $\sigma[a]_\gamma{\sim}_{I^\delta}\sigma[b]_\gamma$ and using again Lemma 4.7, we see that the latter is equivalent to $\Theta_\delta(\sigma_r x, \sigma_g y)$, as we wished to show. For the latter equivalence notice that if we write $\sigma[a]_\gamma = [a']_\delta$ for some $a' \in \mathcal{O}_w$, then $a' + \mathcal{M}_w^\delta = \sigma_r x$ by definition of $\sigma_r = \sigma_{w,\delta} \circ \bar{\sigma} \circ \sigma_{v,\gamma}^{-1}$ (cf. Proposition 3.15).

At this point we have proved the following result.

Theorem 4.10. *Let (K, v) and (L, w) be valued fields and take $\gamma \in vK_{\geq 0}$ and $\delta \in wL_{\geq 0}$. If $(\mathcal{H}_\gamma(K), v_\gamma) \simeq (\mathcal{H}_\delta(L), w_\delta)$, then $K_\gamma \simeq L_\delta$.*

The observation in the following remark will be useful later in Chapter 5.

Remark 4.11. Let (L, w) and (F, u) be valued fields with a common valued subfield (K, v) and take $\gamma \in vK_{\geq 0}$. One then sees the amc-structure of level γ of K as a substructure of the amc-structures of level γ of L and F in a similar way as we do for the γ-valued hyperfields (cf. Remark 3.33 and [26]).

Assume that there is $G \subseteq K^\times/1 + \mathcal{M}_v^\gamma$ and an isomorphism of amc-structures $(\sigma_r, \sigma_g) = \sigma : L_\gamma \to F_\gamma$ such that σ_g fixes the elements of G (i.e., $\sigma_g(x) = x$ for all $x \in G$). Then the isomorphism resulting from Theorem 4.9 fixes the elements of $G \cup \{[0]_\gamma\}$.

Assume now that an isomorphism of valued hyperfields $\sigma : \mathcal{H}_\gamma(L) \to \mathcal{H}_\gamma(F)$ fixes the elements of some subset $H \subseteq \mathcal{H}_\gamma(K)$. We consider the isomorphism of amc-structures (σ_g, σ_r) that we have constructed in the proof of Theorem 4.10. Then σ_g fixes the elements of $H \setminus \{[0]_\gamma\}$. Further, we may write $H = \mathcal{H}_\gamma(A)$ for some $A \subseteq K$. We have that $\bar{\sigma}$ fixes the elements $[a]_{\gamma,\mathcal{H}_\gamma(\mathcal{M}_v^\gamma)}$ of $\mathcal{H}_\gamma(\mathcal{O}_v)/\mathcal{H}_\gamma(\mathcal{M}_v^\gamma)$ with $a \in A$. It follows (cf. Proposition 3.15) that $\sigma_r = \sigma_{u,\gamma} \circ \bar{\sigma} \circ \sigma_{w,\gamma}^{-1}$ fixes the elements $a + \mathcal{M}_v^\gamma$ of $\mathcal{O}_v^\gamma$ with $a \in A$.

4.3 Angular component maps

Definition 4.12. Let (K, v) be a valued field. A map $\alpha : K \to Kv$ is called an *angular component map for* (K, v) if it satisfies the following conditions:

(AC1) $\alpha(x) = 0$ if and only if $x = 0$;

(AC2) $\alpha^\times := \alpha\restriction K^\times$ is a (multiplicative) group homomorphism;

(AC3) $\alpha(x) = xv$ for all $x \in \mathcal{O}_v^\times$.

Remark 4.13. Not all valued fields admit an angular component map. An example of a valued field which does not admit an angular component map is treated in [42].

Example 4.14. Consider $\mathbb{Q}((t))$ with the t-adic valuation v. The map α which sends a Laurent series to its first (in the natural order of $\mathbb{Z}$) non-zero coefficient is an angular component map. In symbols, let

$$x = \sum a_i t^i$$

denote any non-zero element of $\mathbb{Q}((t))$. Then one sets $\alpha(x) := a_{vx}$ and $\alpha(0) := 0$.

Lemma 4.15. *Let (K, v) be a valued field which admits an angular component map α. Then*

$$\mathcal{O}_v^\times \cap \ker \alpha^\times = 1 + \mathcal{M}_v.$$

Proof. If $a \in 1+\mathcal{M}_v \subseteq \mathcal{O}_v^\times$, then by (AC3) $\alpha(a) = av = 1$, whence $a \in \mathcal{O}_v^\times \cap \ker \alpha^\times$. Conversely, assume that $\alpha(a) = 1$ with $va = 0$. This implies that $1 = \alpha(a) = av$ by (AC3) and then $v(a - 1) > 0$. Hence, $a \in 1 + \mathcal{M}_v$. □

Lemma 4.16. *Let (K, v) be a valued field which admits an angular component map α. If $vx < vy$, then $\alpha(x + y) = \alpha(x)$.*

Proof. Since $v(yx^{-1}) > 0$, by (AC2) we have that

$$\alpha(x + y) = \alpha(x)\alpha(1 + yx^{-1}) = \alpha(x),$$

where we used the previous lemma to conclude that $\alpha(1 + yx^{-1}) = 1$. □

Let (K, v) be a valued field. We now wish to investigate some properties of $\mathcal{H}_0(K)$ under the assumption that (K, v) admits an angular component map. We start with the following observation.

Proposition 4.17. *Let (K, v) be a valued field which admits an angular component map α. Take $x, y \in K$ such that $vx \leq vy$. Then*

$$[x]_0 + [y]_0 = \{[z]_0 \in \mathcal{H}_0(K) \mid \alpha(z - y) = \alpha(x) \wedge v(z - y) = vx\}.$$

Proof. We begin by showing the inclusion "$\subseteq$". Assume that $[z]_0 \in [x]_0 + [y]_0$. If $x = y = 0$, then $z = 0$, so $v(z-y) = \infty = vx$ and $\alpha(z-y) = 0 = \alpha(x)$. Otherwise, by Part 2) of Lemma 3.3, we have that

$$v(z - x - y) > vx$$

Thus,

$$\alpha(z - y) = \alpha(z - x - y + x) = \alpha(x),$$

where we used Lemma 4.16. Moreover,

$$v(z - y) = v(z - x - y + x) = \min\{v(z - x - y), vx\} = vx.$$

For the converse inclusion, assume that $\alpha(z-y) = \alpha(x)$ and $v(z-y) = vx$. If $x = 0$, then by assumption $y = 0$ and thus also $z = 0$. We then have $[z]_0 \in [x]_0+[y]_0$ trivially. If $x \neq 0$, then our assumptions imply that $(z - y)x^{-1} \in \mathcal{O}_v^\times \cap \ker \alpha^\times$. Therefore, by Lemma 4.15

$$0 < v((z - y)x^{-1} - 1) = v(zx^{-1} - yx^{-1} - 1).$$

Summing vx to both sides, we obtain

$$vx < v(z - x - y)$$

which by Part 2) of Lemma 3.3 implies that $[z]_0 \in [x]_0 + [y]_0$. This completes the proof. $\square$

With this result we have expressed the fact that $[z]_0 \in [x]_0 + [y]_0$ holds in $\mathcal{H}_0(K)$ by means of two equalities: one in the residue field Kv and one in the value group vK.

We will now prove some other useful lemmas about angular component maps.

Lemma 4.18. *Let (K, v) be a valued field which admits an angular component map α. If $x, y \in K$ are such that $v(x-y) = vx = vy$, then $\alpha(x-y) = \alpha(x)-\alpha(y)$.*

Proof. If $x = y = 0$, then there is nothing to prove. Since by assumption $vx = vy$, we may assume that $x, y \in K^\times$.

We have that $\alpha(x - y) = \alpha(x)\alpha(1 - yx^{-1})$ by (AC2). Now since $vx = v(x - y)$ we have that $v(1 - yx^{-1}) + vx = vx$, therefore $1 - yx^{-1} \in \mathcal{O}_v^\times$. By (AC3) we then obtain

$$\alpha(x - y) = \alpha(x)(1 - yx^{-1})v.$$

Now observe that $vx = vy$ so that $yx^{-1} \in \mathcal{O}_v^\times$. Since $1 \in \mathcal{O}_v^\times$, we obtain that

$$\alpha(x - y) = \alpha(x)(1v - (yx^{-1})v).$$

Applying (AC2) and (AC3) we then have that

$$\alpha(x - y) = \alpha(x)(\alpha(1) - \alpha(yx^{-1})) = \alpha(x) - \alpha(y).$$ $\square$

Lemma 4.19. *Let (K, v) be a valued field which admits an angular component map α. Then $\alpha(-x) = -\alpha(x)$ for all $x \in K$.*

Proof. By (AC2) it suffices to show that $\alpha(-1) = -1v$. This is true by (AC3) since $-1 \in \mathcal{O}_v^\times$. □

Let (K, v) be a valued field. Observe that we have a short exact sequence of abelian groups

$$\{1\} \longrightarrow (Kv)^\times \xrightarrow{\iota} \mathcal{H}_0(K)^\times \xrightarrow{v_0} vK \longrightarrow \{0\} \tag{4.5}$$

where $\iota(xv) := [x]_0$ for all $xv \in (Kv)^\times$. This is well-defined and injective since if $vx = vy = 0$, then

$$v(x - y) > 0 \iff v(1 - xy^{-1}) > 0.$$

Clearly, it is a homomorphism of (multiplicative) abelian groups. Moreover,

$$\operatorname{Im}\iota = \mathcal{H}_0(\mathcal{O}_v^\times) = \ker v_0 \ .$$

Proposition 4.20. *If (K, v) is a valued field which admits an angular component map α, then the exact sequence (4.5) splits.*

Proof. We define $\bar{\alpha} : \mathcal{H}_0(K)^\times \to (Kv)^\times$ setting $\bar{\alpha}[x]_0 := \alpha(x)$ for all $[x]_0 \in \mathcal{H}_0(K)^\times$. Assume that $[x]_0 = [y]_0$. Then $xy^{-1} \in 1 + \mathcal{M}_v \subseteq \ker\alpha^\times$ (cf. Lemma 4.15). Now by (AC2) we obtain that $1 = \alpha(xy^{-1}) = \alpha(x)\alpha(y)^{-1}$ and therefore $\alpha(x) = \alpha(y)$. This shows that $\bar{\alpha}$ is well-defined. For all $x \in \mathcal{O}_v^\times$ we have that $\alpha(x) = xv$ by (AC3). This implies that $\bar{\alpha}$ is a left split of (4.5). □

Proposition 4.21. *If (K, v) is a valued field which admits an angular component map α, then*

$$\begin{aligned} h : \mathcal{H}_0(K)^\times &\to (Kv)^\times \oplus vK \\ [x]_0 &\mapsto (\alpha(x), vx) \end{aligned}$$

is an isomorphism of abelian groups.

Proof. It is well-defined since if $[x]_0 = [y]_0$, then $vx = vy$ by Proposition 3.1 and

$$\alpha(x) = \alpha(xy^{-1})\alpha(y) = \alpha(y)$$

follows from the fact that $xy^{-1} \in 1 + \mathcal{M}_v \subseteq \ker\alpha^\times$ (Lemma 4.15) and (AC2).

To see that it is injective assume that $(\alpha(x), vx) = (\alpha(y), vy)$ for some $x, y \in K^\times$. By Lemma 4.15 and our assumptions we obtain that $yx^{-1} \in 1 + \mathcal{M}_v$. Therefore, that $[x]_0 = [x(yx^{-1})]_0 = [y]_0$. Indeed, $[x]_0$ denotes the multiplicative coset $x(1 + \mathcal{M}_v)$.

In order to prove surjectivity we take $(xv, vy) \in (Kv)^\times \oplus vK$, where $x \in \mathcal{O}_v^\times$ and $y \in K^\times$. Let $a \in \mathcal{O}_v^\times$ be such that $\alpha(y) = av$. Then

$$\alpha(ya^{-1}) = \alpha(y)\alpha(a)^{-1} = 1.$$

We consider $[xya^{-1}]_0 \in \mathcal{H}_0(K)^\times$. We have that

$$h[xya^{-1}]_0 = (\alpha(xya^{-1}), v(xya^{-1})) = (\alpha(x)\alpha(ya^{-1}), vy{+}v(xa^{-1})) = (\alpha(x), vy) = (xv, vy),$$

where we have used the fact that $va = vx = 0$ and (AC3).

Finally, take $[x]_0, [y]_0 \in \mathcal{H}_0(K)^\times$. We obtain that

$$h[x]_0[y]_0 = h[xy]_0 = (\alpha(xy), v(xy)) = (\alpha(x)\alpha(y), vx + vy).$$

Moreover,

$$h[1]_0 = (\alpha(1), v1) = (1v, 0)$$

and

$$h[x]_0^{-1} = h[x^{-1}]_0 = (\alpha(x^{-1}), vx^{-1}) = (\alpha(x)^{-1}, -vx).$$

This shows that h is an isomorphism of abelian groups. □

If we extend h to $\mathcal{H}_0(K)$ by setting $h([0]_0) := (0, \infty)$, then we obtain a bijective function

$$\mathcal{H}_0(K) \to ((Kv)^\times \times vK) \cup \{(0, \infty)\}$$

which we still denote by h.

It is easy to see that we can apply Lemma 2.15 in this situation. We obtain a hyperring with underlying set $((Kv)^\times \times vK) \cup \{(0, \infty)\}$. Let us denote this hyperring by $Kv \oplus (vK \cup \{\infty\})$. It clearly is a hyperfield. In the following theorem we collect what we have proved.

Theorem 4.22. *Let (K, v) be a valued field which admits an angular component map α. Then $\mathcal{H}_0(K)$ is isomorphic as a hyperfield to $Kv \oplus (vK \cup \{\infty\})$ under the isomorphism defined by $h[x]_0 := (\alpha(x), vx)$.*

4.4 Graded rings and anneids

Let us consider rings A with unity $1 \in A$. If $X, Y \in \mathcal{P}^*(A)$, then by XY we denote the set of finite sums in A of elements of the form xy for $x \in X$ and $y \in Y$.

Definition 4.23. Let Γ be a group, denoted additively. We say that a ring A is Γ-*graded* if there is a family $\{A_\gamma \mid \gamma \in \Gamma\}$ of subgroups of $(A, +, 0)$ such that

(GR1) $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ as abelian groups, (GR2) $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$.

We call Γ the *grading group* of A. The set $H(A) := \bigcup_{\gamma\in\Gamma} A_\gamma$ is called the *set of homogeneous elements of* A and, for $\gamma \in \Gamma$, a nonzero $x \in A_\gamma$ is said to be *homogeneous of degree* γ. The additive identity 0 of A is considered as homogeneous but it has no degree.

For more information about general graded rings we refer the reader to [39].

We are interested in a special kind of graded rings which are naturally associated to valued fields.

Let (K, v) be a valued field. For any $\gamma \in vK$ we consider

$$\mathcal{P}^\gamma := \{x \in K \mid vx \geq \gamma\}$$

and

$$\mathcal{M}^\gamma := \{x \in K \mid vx > \gamma\}.$$

Note that for all $\gamma \in vK$ we have that $(\mathcal{P}^\gamma, +, 0)$ is an abelian group and that $\mathcal{M}^\gamma$ is a subgroup of it.

Definition 4.24. Define the *graded ring of* (K, v) to be

$$\mathrm{gr}_v(K) := \bigoplus_{\gamma\in vK} \mathcal{P}^\gamma/\mathcal{M}^\gamma.$$

We set $\mathrm{in}_v(x) := x + \mathcal{M}^{vx} \in \mathcal{P}^{vx}/\mathcal{M}^{vx} \subseteq \mathrm{gr}_v(K)$ for $x \in K^\times$. The homogeneous element $\mathrm{in}_v(x)$ is called the *initial form* of $x \in K^\times$. By convention, $\mathrm{in}_v(0)$ will be understood as the zero of this graded ring.

For the convenience of the reader, let us state the following straightforward observation.

Lemma 4.25. *Let (K, v) be a valued field and let $x, y \in K^\times$. If $v(x+y) = vx = vy$, then the sum of $\mathrm{in}_v(x)$ and $\mathrm{in}_v(y)$ in $\mathrm{gr}_v(K)$ is the homogeneous element $\mathrm{in}_v(x+y)$.*

Proof. Since $vx = vy$, we have that the sum in $\mathrm{gr}_v(K)$ of $\mathrm{in}_v(x) = x + \mathcal{M}^{vx}$ and $\mathrm{in}_v(y) = y + \mathcal{M}^{vy}$ is $(x + y) + \mathcal{M}^{vx}$. Indeed, $\mathrm{in}_v(x)$ and $\mathrm{in}_v(y)$ belong to the same subgroup $\mathcal{P}^{vx}/\mathcal{M}^{vx}$ of $\mathrm{gr}_v(K)$ and therefore their sum in $\mathrm{gr}_v(K)$ corresponds to their sum in this subgroup. Now, the assumption $v(x + y) = vx$ yields

$$(x + y) + \mathcal{M}^{vx} = (x + y) + \mathcal{M}^{v(x+y)} = \mathrm{in}_v(x + y). \qquad \square$$

Let (K, v) be a valued field. Recall that for $x \in K^\times$ we denote by $[x]_0$ the class of x in $\mathcal{H}_0(K)$, which is the multiplicative coset $x(1 + \mathcal{M})$.

Lemma 4.26. *Let (K, v) be a valued field. For all $x \in K^\times$*

$$\mathrm{in}_v(x) = [x]_0.$$

as subsets of K. In particular, for all $x, y \in K$ if $\mathrm{in}_v(x) = \mathrm{in}_v(y)$, then $vx = vy$.

Proof. By Part 1) of Lemma 3.3 we have that

$$[x]_0 = \{a \in K \mid v(x - a) > vx\} = x + \mathcal{M}^{vx} = \mathrm{in}_v(x).$$

Hence, for $x, y \in K^\times$, if $\mathrm{in}_v(x) = \mathrm{in}_v(y)$, then $[x]_0 = [y]_0$ and $vx = vy$ follows from the fact that v_0 is well-defined on $\mathcal{H}_0(K)$ as we have shown in the proof of Proposition 3.1. On the other hand, since if $\mathrm{in}_v(x)$ is the zero of $\mathrm{gr}_v(K)$, then $x = 0$, we obtain that if $x = 0$ or $y = 0$, then also the other is 0 and therefore, in this case, $vx = \infty = vy$. □

Remark 4.27. The previous result shows that if we identify the respective zeros, then $\mathcal{H}_0(K)$ and $H(\mathrm{gr}_v(K))$ are exactly the same family H of subsets of K. Now we run into a problem of notation that we want to address in this remark.

On H we have defined the hyperoperation which we denote by $+$. On the other hand, if we regard H as the subset of the homogeneous elements of $\mathrm{gr}_v(K)$, then we have also the addition of $\mathrm{gr}_v(K)$. To avoid confusion we shall always denote the latter operation by $\oplus$. Therefore, if $x, y \in K$ we shall denote by $\mathrm{in}_v(x) \oplus \mathrm{in}_v(y)$ the element of $\mathrm{gr}_v(K)$ which is represented as the sum of the homogeneous elements $\mathrm{in}_v(x)$ and $\mathrm{in}_v(y)$. More generally, for $a, b \in \mathrm{gr}_v(K)$ we will denote their sum in $\mathrm{gr}_v(K)$ as $a \oplus b$. On the other hand, the symbol $+$ will be reserved for the hyperoperation of $\mathcal{H}_0(K)$ as we have done until now.

Remark 4.28. The function $\mathrm{in}_v : K \to H(\mathrm{gr}_v(K))$ is surjective.

Moreover, for $a \in \mathrm{gr}_v(K) \setminus \{0\}$, by (GR1), we have a unique representation

$$a = \mathrm{in}_v(x_1) \oplus \mathrm{in}_v(x_2) \oplus \ldots \oplus \mathrm{in}_v(x_n), \tag{4.6}$$

where $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in K^\times$ with $vx_1 < vx_2 < \ldots < vx_n$.

Remark 4.29. On nonzero homogeneous elements of $\mathrm{gr}_v(K)$ a product can be defined as

$$\mathrm{in}_v(x) \cdot \mathrm{in}_v(y) := \mathrm{in}_v(xy). \tag{4.7}$$

By Lemma 4.26 this operation is well-defined as it corresponds to the product in $\mathcal{H}_0(K)^\times$. Moreover, the set of nonzero homogeneous elements forms an abelian group under this operation with identity $\mathrm{in}_v(1) = [1]_0$. We can extend this product to the whole set of homogeneous elements by setting the zero as an absorbing element. Note that this is in agreement with our convention that $\mathrm{in}_v(0)$ is the zero of the graded ring. One can then extend this operation, defined on homogeneous elements, to $\mathrm{gr}_v(K)$ distributing over $\oplus$. We call the product in $\mathrm{gr}_v(K)$ thus obtained *induced* from the product of homogeneous elements. With this product, $\mathrm{gr}_v(K)$ is a vK-graded ring (property (GR2) is straightforward to verify).

Let (K, v) be a valued field. The graded ring $\mathrm{gr}_v(K)$ satisfies the following two additional properties:

(GR3) the grading group is an ordered abelian group;

(GR4) the set of nonzero homogeneous elements is a group under multiplication.

Warning: From now on, for $x \in K$, we will use both notations $[x]_0$ and $\mathrm{in}_v(x)$ according to what is more convenient for the argument we are presenting. We will frequently switch from one notation to the other without referring to Lemma 4.26.

In what follows we will study some more properties of the graded ring $\mathrm{gr}_v(K)$ of some valued field (K, v) which link it to the 0-valued hyperfield $\mathcal{H}_0(K)$. In particular, we are interested in finding a way to express that $[z]_0 \in [x]_0 + [y]_0$ for some $x, y, z \in K$, using the homogeneous elements $\mathrm{in}_v(x), \mathrm{in}_v(y)$ and $\mathrm{in}_v(z)$ and the operation $\oplus$ of the graded ring. This will be achieved in Proposition 4.33 below. The following concept will play a crucial role.

Definition 4.30. If $a \in \mathrm{gr}_v(K)$, then we define $g(a)$ to be the homogeneous element of minimal degree in the unique representation (4.6) of a.

This defines a function

$$g : \mathrm{gr}_v(K) \to H(\mathrm{gr}_v(K))$$

which is called the *initial form function of* $\mathrm{gr}_v(K)$.

Clearly the initial form function is the identity on homogeneous elements and is surjective. Moreover, it can be interpreted as a map from the (hyper)ring $\mathrm{gr}_v(K)$ (cf. Remark 2.4) onto the hyperfield $\mathcal{H}_0(K)$. With this interpretation in mind, we state and prove the following result.

Lemma 4.31. *The initial form function g is a homomorphism of hyperrings.*

Proof. The initial form function g satisfies (HH1) by definition.

In order to show (HH2) we note that, by the definition of the product in $\mathrm{gr}_v(K)$, the homogeneous element of minimal degree $g(ab)$ in the representation (4.6) of ab is $[x_1y_1]_0$. Therefore,

$$g(ab) = [x_1y_1]_0 = [x_1]_0[y_1]_0 = g(a)g(b).$$

Let us now take nonzero $a, b \in \mathrm{gr}_v(K)$ and let $x_1, \ldots, x_n, y_1, \ldots, y_m \in K^\times$ be such that $vx_1 < \ldots < vx_n$, $vy_1 < \ldots < vy_m$ and

$$a = \mathrm{in}_v(x_1) \oplus \mathrm{in}_v(x_2) \oplus \ldots \oplus \mathrm{in}_v(x_n) \text{ and}$$
$$b = \mathrm{in}_v(y_1) \oplus \mathrm{in}_v(x_2) \oplus \ldots \oplus \mathrm{in}_v(y_m).$$

Then, by definition, $g(a) = [x_1]_0$ and $g(b) = [y_1]_0$. In order to show that g satisfies (HH3) we have to prove that

$$g(a \oplus b) \in [x_1]_0 + [y_1]_0 \ . \tag{4.8}$$

If $vx_1 < vy_1$, then by Part 2) of Lemma 3.3 we obtain that $[x_1]_0 \in [x_1]_0 + [y_1]_0$. However, in this case we have $g(a \oplus b) = [x_1]_0$ so (4.8) holds. If $vy_1 < vx_1$ one can proceed in a similar way.

It remains to show that (4.8) holds in the case $vx_1 = vy_1$. At this point let us distinguish two cases:

- If $v(x_1+y_1) = vx_1$, then by Lemma 4.25 the homogeneous element of minimal degree in the representation of $a \oplus b$ is

  $$g(a \oplus b) = \mathrm{in}_v(x_1 + y_1) = [x_1 + y_1]_0 \ .$$

  and by Part 4) of Lemma 3.3 we conclude that $g(a \oplus b) \in [x_1]_0 + [y_1]_0$.

- If $v(x_1+y_1) > vx_1$, then by Part 1) of Lemma 3.3 we obtain that $[y_1] = -[x]_1$ and by Part 2) of Lemma 3.3 we have that

  $$[x_1]_0 - [x_1]_0 = \{[z]_0 \in \mathcal{H}_0(K) \mid vz > vx_1\}. \tag{4.9}$$

  In this case $\mathrm{in}_v(x_1) \oplus \mathrm{in}_v(y_1)$ is zero in $\mathrm{gr}_v(K)$. If $a \oplus b$ is zero in $\mathrm{gr}_v(K)$, then $g(a \oplus b) \in [x_1]_0 - [x_1]_0$, so we are done. If $a \oplus b$ is not zero, then analyzing the representation of $a \oplus b$, we see that the homogeneous element of minimal degree $g(a \oplus b)$ must now have the form $\mathrm{in}_v(x_i)$ or $\mathrm{in}_v(y_j)$ or $\mathrm{in}_v(x_i + y_j)$ for some $2 \le i \le n$ and $2 \le j \le m$. Note that if the last case occurs, then $v(x_i + y_j) = vx_i = vy_j$ must hold.
  It now suffices to observe that by assumption, for $2 \le i \le n$ and $2 \le j \le m$ we have that $vx_i > vx_1$ and $vy_j > vy_1 = vx_1$. Thus, if we set $[z]_0 := g(a \oplus b)$, then we will have that $vz > vx_1$ in all of the three cases mentioned above. We can then conclude that $[z]_0 \in [x_1]_0 - [x_1]_0$ by (4.9).

We have proved that $g(\{a \oplus b\}) \subseteq g(a) + g(b)$ for nonzero $a, b \in \mathrm{gr}_v(K)$. Clearly, this implies that g satisfies (HH3) (cf. Remark 2.4). □

Remark 4.32. The initial form function is another example of a homomorphism of hyperrings with trivial kernel which is not injective (cf. Remark 3.35).

In the next proposition, we use the symbol $\ominus$ to denote the minus sign of the ring $\mathrm{gr}_v(K)$ (cf. Remark 4.27). Thus, as usual, $a \ominus b$ will denote the sum of a with the additive inverse of b in $gr_v(K)$. Note that if $a = \mathrm{in}_v(x)$ is an homogeneous element of $\mathrm{gr}_v(K)$, then its additive inverse in $\mathrm{gr}_v(K)$ is $\mathrm{in}_v(-x)$. This implies that the additive inverse of $\mathrm{in}_v(x)$ in $\mathrm{gr}_v(K)$ and the hyperadditive inverse of $[x]_0$ in $\mathcal{H}_0(K)$ coincide with $[-x]_0 = \mathrm{in}_v(-x)$.

Proposition 4.33. *Let (K, v) be a valued field. Take $x, y \in K$ with $vx \leq vy$. Then*

$$[x]_0 + [y]_0 = \{[z]_0 \in \mathcal{H}_0(K) \mid g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(y)) = \mathrm{in}_v(x)\}.$$

Proof. We first prove the inclusion "$\subseteq$". For take $[z]_0 \in [x]_0 + [y]_0$. If x and y are both 0, then $[z]_0 = [0]_0$, so

$$g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(y)) = 0 = \mathrm{in}_v(x).$$

Hence, the inclusion "$\subseteq$" holds in this case. We can then assume without loss of generality that x and y are not both 0.

We now compute $g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(y))$ in two different cases.

- If $[0]_0 \notin [x]_0 + [y]_0$, then $[z]_0 = [x + y]_0$ by Lemma 3.6 and $v(x + y) = vx$ because if $v(x + y) > vx$, then by Part 2) of Lemma 3.3 we would obtain $[0]_0 \in [x]_0 + [y]_0$. Hence in this case,

  $$\mathrm{in}_v(z) \ominus \mathrm{in}_v(y) = \mathrm{in}_v(x + y) \ominus \mathrm{in}_v(y).$$

  Now, if $vy = v(x+y) = vx$, then $\mathrm{in}_v(x+y)\ominus\mathrm{in}_v(y) = \mathrm{in}_v(x+y-y) = \mathrm{in}_v(x)$, so $g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(y)) = \mathrm{in}_v(x)$. Otherwise, $vy > vx = v(x + y)$, whence $g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(y)) = x + y + \mathcal{M}^{vx} = \mathrm{in}_v(x)$.

- If $[0]_0 \in [x]_0 + [y]_0$, then $[y]_0 = -[x]_0$ and by Part 2) of Lemma 3.3, $[z]_0 \in [x]_0 - [x]_0$ means that $vz > vx$. Thus, since $-[x]_0 = [-x]_0 = \mathrm{in}_v(-x)$, we obtain that

  $$g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(y)) = g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(-x)) = g(\mathrm{in}_v(z) \oplus \mathrm{in}_v(x)) = \mathrm{in}_v(x).$$

  by the definition of g.

For the converse inclusion, we now assume that $g(\mathrm{in}_v(z)\ominus\mathrm{in}_v(y)) = \mathrm{in}_v(x)$ for some $x, y, z \in K$ with $vx \leq vy$. We have to show that $[z]_0 \in [x]_0 + [y]_0$. Recall that, by definition, g is the identity on homogeneous elements. Hence, by the previous lemma we obtain that

$$[x]_0 = \mathrm{in}_v(x) = g(\mathrm{in}_v(z) \ominus \mathrm{in}_v(y)) \in [z]_0 - [y]_0.$$

Now, $[z]_0 \in [x]_0 + [y]_0$ follows by the reversibility axiom (H4). $\square$

Remark 4.34. Define the following operation on the set $H(\mathrm{gr}_v(K))$ of homogeneous elements of $\mathrm{gr}_v(K)$:

$$\mathrm{in}_v(x) \circ \mathrm{in}_v(y) := g(\mathrm{in}_v(x) \oplus \mathrm{in}_v(y)) \quad (x, y \in K).$$

Then $\circ$ is the same as the operation $*$ that we introduced on $\mathcal{H}_0(K)$ in Section 3.2.

To see this we take $x, y \in K$ and assume without loss of generality that $vx \leq vy$. We distinguish two cases.

- If $[0]_0 \notin [x]_0 + [y]_0$, then $v(x+y) = vx$ as we observed in the proof of Lemma 3.6. Now, in the case $vx < vy$ we obtain $\mathrm{in}_v(x) \circ \mathrm{in}_v(y) = \mathrm{in}_v(x)$. On the other hand, in this case $[x]_0 * [y]_0 = [x+y]_0 = [x]_0$, by Part 1) of Lemma 3.3. Therefore,

  $$[x]_0 * [y]_0 = [x]_0 = \mathrm{in}_v(x) = \mathrm{in}_v(x) \circ \mathrm{in}_v(y),$$

  In the case $vx = vy$, since $v(x+y) = vx$, we can apply Lemma 4.25 to obtain that

  $$\mathrm{in}_v(x) \circ \mathrm{in}_v(y) = \mathrm{in}_v(x+y) = [x+y]_0 = [x]_0 * [y]_0 \ .$$

  where we have used the fact that g is the identity on homogeneous elements.

- If $[0]_0 \in [x]_0 + [y]_0$, then $[y]_0 = -[x]_0$. Therefore, $vx = vy$ and

  $$\mathrm{in}_v(x) \oplus \mathrm{in}_v(y) = \mathrm{in}_v(x) \ominus \mathrm{in}_v(x)$$

  is zero in $\mathrm{gr}_v(K)$. Thus, $\mathrm{in}_v(x) \circ \mathrm{in}_v(y) = \mathrm{in}_v(0) = [0]_0 = [x]_0 * [y]_0$.

Let us add to this section the following simple observation which we see as an analog of Proposition 4.17 with initial forms.

Proposition 4.35. *Let (K, v) be a valued field. Take $x, y \in K$ with $vx \leq vy$. Then*

$$[x]_0 + [y]_0 = \{[z]_0 \in \mathcal{H}_0(K) \mid \mathrm{in}_v(z-y) = \mathrm{in}_v(x)\}.$$

Proof. We first show the inclusion "$\subseteq$". Assume that $[z]_0 \in [x]_0 + [y]_0$. If $x = y = 0$, then $z = 0$, whence $\mathrm{in}_v(z-y) = 0 = \mathrm{in}_v(x)$ and so the inclusion holds. We can then assume without loss of generality that x and y are not both 0. By Part 2) of Lemma 3.3 we have that $v(z-x-y) > vx$. Thus,

$$v(z-y) = v(z-x-y+x) = \min\{v(z-x-y), vx\} = vx.$$

Now, $\mathrm{in}_v(z-y) = \mathrm{in}_v(x)$ holds if and only if $z-y-x \in \mathcal{M}^{vx} = \mathcal{M}^{v(y-z)}$ and this follows from $v(z-x-y) > vx$. Therefore, $\mathrm{in}_v(z-y) = \mathrm{in}_v(x)$.

For the converse inclusion, we assume that $\mathrm{in}_v(z-y) = \mathrm{in}_v(x)$ for some $x, y, z \in K$ with $vx \leq vy$. By Part 4) of Lemma 3.3 we have that

$$[x]_0 = [z-y]_0 \in [z]_0 - [y]_0.$$

Thus, $[z]_0 \in [x]_0 + [y]_0$ follows by the reversibility axiom (H4). □

We will now express the hyperoperation of $\mathcal{H}_0(K)$ using just the group additions of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ for $\gamma \in vK$, thus avoiding the use of the initial form function. For this we have to introduce a construction which appears in [4, Definition 1.5].

We wish to define the wedge sum of a family of canonical hypergroups $\{F_g \mid g \in G\}$ indexed by a totally ordered set $(G, <)$. For convenience, we present this construction with the order of G reversed with respect to the order considered in [4]. We assume that the F_g are disjoint but we identify their neutral elements with one element 0 which then belongs to each F_g . We denote the hyperoperation of F_g by $+_g$. Let F be the union of the F_g over all $g \in G$. Further, let ψ denote the surjective function $F \setminus \{0\} \to G$ sending a nonzero element of F_g to g.

A hyperoperation $\boxplus$ on F is defined as follows: $x \boxplus 0 = 0 \boxplus x := \{x\}$ for all $x \in F$ and, for $x, y \in F \setminus \{0\}$,

$$x \boxplus y := \begin{cases} \{x\} & \text{if } \psi(x) < \psi(y) \\ \{y\} & \text{if } \psi(y) < \psi(x) \\ x +_{\psi(x)} y & \text{if } \psi(x) = \psi(y) \text{ and } 0 \notin x +_{\psi(x)} y \\ (x +_{\psi(x)} y) \cup \bigcup_{g > \psi(x)} F_g & \text{if } \psi(x) = \psi(y) \text{ and } 0 \in x +_{\psi(x)} y \end{cases} \tag{4.10}$$

This gives to F the structure of a canonical hypergroup as proved in [4, Lemma 3.1]. The canonical hypergroup thus obtained is called the *wedge sum* of $\{F_g \mid g \in G\}$ and is denoted by $\bigvee_{g \in G} F_g$.

Let (K, v) be a valued field. Recall that the union of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ over all $\gamma \in vK$, i.e., the set of homogeneous elements of $\mathrm{gr}_v(K)$, is the same set as $\mathcal{H}_0(K)$. On $\mathcal{H}_0(K)$ we have defined a hyperoperation $+$, that is the hyperoperation of the factor hyperfield of K modulo its multiplicative subgroup $1 + \mathcal{M}_v$.

We are now going to show that this hyperoperation and the hyperoperation $\boxplus$ obtained on $\mathcal{H}_0(K)$ with the wedge sum construction of the (hyper)groups $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ for $\gamma \in vK$, are the same thing. According to the definition of the wedge sum given above for $\gamma \in vK$ we will denote by $+_\gamma$ the hyperoperation of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$. Note that this hyperoperation always results in a singleton, since $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ is an abelian group for all $\gamma \in vK$ (cf. Remark 2.4).

In this situation, ψ is just the degree map of $\mathrm{gr}_v(K)$ and by definition the degree of the homogeneous element $\mathrm{in}_v(x)$ is vx. Thus, the function ψ in this setting is the valuation v_0 of $\mathcal{H}_0(K)$ sending $[x]_0$ to vx for all $x \in K^\times$.

Let us pick two elements of $[x]_0, [y]_0 \in \mathcal{H}_0(K)^\times$. Assume that $v_0[x]_0 < v_0[y]_0$. Then $[x]_0 \boxplus [y]_0 = \{[x]_0\}$ according to the definition of the wedge sum. On the other hand, in this case we know that $[x]_0 + [y]_0 = \{[x]_0\}$ from Part 2) of Lemma 3.3 and Lemma 3.6.

If $v_0[y]_0 < v_0[x]_0$, then we obtain $[x]_0 \boxplus [y]_0 = [x]_0 + [y]_0$ analogously.

Assume now that $v_0[x]_0 = v_0[y]_0 =: \gamma$ and that $\mathrm{in}_v(x) +_\gamma \mathrm{in}_v(y)$ does not contain the zero of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$. This implies that $v(x+y)=\gamma$, since

$$\mathrm{in}_v(x)+_\gamma \mathrm{in}_v(y)=\{(x+y)+\mathcal{M}^\gamma\}.$$

Hence, by Lemma 4.25 we obtain that

$$[x]_0 \boxplus [y]_0 = \mathrm{in}_v(x)+_\gamma \mathrm{in}_v(y)=\{\mathrm{in}_v(x+y)\}=\{[x+y]_0\}.$$

On the other hand, since $v(x+y)=vx=vy$, we obtain that $[0]_0\notin [x]_0+[y]_0$ by Part 2) of Lemma 3.3 and therefore $[x]_0+[y]_0=\{[x+y]_0\}$ by Lemma 3.6.

Finally, assume that $v_0[x]_0=v_0[y]_0=:\gamma$ and that $\mathrm{in}_v(x)+_\gamma \mathrm{in}_v(y)$ contains the zero of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$. This means that, by definition of the wedge sum,

$$[x]_0 \boxplus [y]_0 = ([x]_0 +_\gamma [y]_0)\cup \bigcup_{\delta>\gamma} \mathcal{P}^\delta/\mathcal{M}^\delta.$$

Since by assumption $[x]_0+_\gamma [y]_0$ is the singleton containing only the zero of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ and this zero belongs to $\mathcal{P}^\delta/\mathcal{M}^\delta$ for any $\delta\in vK$, we obtain that

$$[x]_0 \boxplus [y]_0 = \bigcup_{\delta>\gamma} \mathcal{P}^\delta/\mathcal{M}^\delta.$$

This is the set of all homogeneous elements of $\mathrm{gr}_v(K)$ with degree larger than γ, namely,

$$[x]_0 \boxplus [y]_0 = \{[z]_0\in \mathcal{H}_0(K)\mid vz>\gamma\}$$

Since $\gamma=vx$, as we have already observed before (cf. (4.9)) this subset of $\mathcal{H}_0(K)$ is exactly $[x]_0-[x]_0$. Therefore, it remains to show that $[y]_0=-[x]_0$. By Part 1) of Lemma 3.3, this happens if and only if $v(x+y)>vx$; but this follows from the fact that $\mathrm{in}_v(x)+_\gamma \mathrm{in}_v(y)=\{(x+y)+\mathcal{M}^\gamma\}$ contains the zero of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$.

Remark 4.36. Let us recall the short exact sequence of abelian groups (4.5). Following the terminology of [4, Definition 4.1] we see that $\mathcal{H}_0(K)$ is actually the *vK-layering* of Kv along this short exact sequence. We will not use this result later in this text and we will not give a proof of it. This is because the proof requires a lot of notation and is technical in its nature. The interested reader may verify this assertion by looking at [4, Definition 4.1] and noticing that for $\gamma\in vK$ one has that $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ is the pre-image $v_0^{-1}(\gamma)$ under v_0 of γ in $\mathcal{H}_0(K)$, together with $[0]_0$. Moreover, its hyperoperation $+_\gamma$ is obtained from the hyperaddition $+_0$ of $\mathcal{P}^0/\mathcal{M}^0=Kv$ as required in [4, Definition 4.1]. This with the fact that the hyperoperation of $\mathcal{H}_0(K)$ coincides with that of the wedge sum of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ for $\gamma\in vK$, as we have shown above, yields the result.

We would like to conclude this section studying possible first order languages to work with graded rings. The first natural attempt is to consider the language $\mathcal{L}_r$ of rings. Indeed, any graded ring is a ring and thus an $\mathcal{L}_r$-structure. The problem with this approach is that, regardless on how we recognize the homogeneous elements, e.g., with a unary relation symbol, if the grading group is infinite (as in the case of $\mathrm{gr}_v(K)$), then in a saturated[1] model we will always have elements which are not finite sums of homogeneous elements. Therefore, an $\mathcal{L}_r$-theory whose models are exactly graded rings cannot exist. If one is not interested in such a theory, then we propose as the *language of graded rings* (with ordered grading group), the language $\mathcal{L}_r$ of rings extended with a unary function symbol g, to be interpreted as the initial form function. We denote this language by $\mathcal{L}_{gr}$.

However, we would like to investigate another possibility too. In his article [22] Krasner introduced the notion of *anneid* and related it to graded rings. Following his idea, it is possible to think about the following language.

By the *language $\mathcal{L}_{an}$ of anneids* we mean a language with two sorts $\mathbf{H}$ and $\mathbf{G}$. On the sort $\mathbf{G}$ the language $\mathcal{L}_{og}$ of ordered (additive) abelian groups $\{+, -, 0, <\}$ and on the sort $\mathbf{H}$ the language $\mathcal{L}_{zs} := \{Z, S, \cdot, ^{-1}, 1\}$ where Z is a unary relation symbol, S a ternary relation symbol and $\{\cdot, ^{-1}, 1\}$ is the language of multiplicative groups. We further add to $\mathcal{L}_{an}$ a unary function symbol deg of type $(\mathbf{H}, \mathbf{G})$.

Let (K, v) be a valued field. Let us now explain how $\mathrm{gr}_v(K)$ can be viewed as an $\mathcal{L}_{an}$-structure. Actually, $\mathrm{gr}_v(K)$ itself will not be an $\mathcal{L}_{an}$-structure but the pair $(H(\mathrm{gr}_v(K)), vK)$ formed by the set of its homogeneous elements and its grading group will be such a structure. By this we mean that the universe of the sort $\mathbf{G}$ will be vK and the universe of the sort $\mathbf{H}$ will be the set of homogeneous elements of $\mathrm{gr}_v(K)$. We interpret $\{+, -, 0, <\}$ in vK as usual and deg as the degree function. In this setting, we think of each $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ having its own zero, so that its degree is γ and the degree function is defined everywhere. The unary relation symbol Z will encode the set of all these zeros. The ternary relation symbol S will encode all the graphs of the addition in $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ for $\gamma \in vK$, simultaneously. Thus, $S(\mathrm{in}_v(x), \mathrm{in}_v(y), \mathrm{in}_v(z))$ if and only if $vx = vy$ and $\mathrm{in}_v(z)$ is the sum of $\mathrm{in}_v(x)$ and $\mathrm{in}_v(y)$ in $\mathcal{P}^{vx}/\mathcal{M}^{vx}$. Finally, $\{\cdot, ^{-1}, 1\}$ will be used to encode the multiplicative structure of the set of homogeneous elements (cf. Remark 4.29).

We will now give an $\mathcal{L}_{an}$-theory $\mathbf{T}_{gr}$ with the following property. Given a model $(H, G, \deg)$ of $\mathbf{T}_{gr}$,

$$A := \bigoplus_{g \in G} \deg^{-1}(g)$$

with the multiplication induced by that of H, is a G-graded ring. We call the graded ring thus obtained the graded ring *associated* to the model $(H, G, \deg)$ of

[1]We have not introduced the notion of saturation in this text, since we use it only here. For a definition we refer to [43, Section 2.5].

$\mathbf{T}_{gr}$. Clearly, the set of homogeneous elements of the graded ring associated to a model $(H, G, \deg)$ of $\mathbf{T}_{gr}$ is H. The induced product on A is then defined as we have explained in Remark 4.29.

In listing the axioms of this theory we will use $x, y, z, \ldots$ for variables of the sort $\mathbf{H}$ and $g, h, \ldots$ for variables of the sort $\mathbf{G}$. In this way it will always be clear over which sort we quantify. As usual, for a formula $\varphi(x)$ with one free variable x, we write $\exists! x\varphi(x)$ to abbreviate $\exists x(\varphi(x) \wedge \forall y(\varphi(y) \rightarrow x = y))$.

(OAG) The $\mathcal{L}_{og}$-axioms for ordered abelian groups.

(AGR) The $\mathcal{L}_{zs}$-axioms of the form $\forall(\neg Z(\underline{x}) \rightarrow \varphi(\underline{x}))$ where $\underline{x}$ is a tuple of variables of appropriate length and $\varphi(\underline{x})$ is an axiom of the theory of (multiplicative) abelian groups. We further add the axiom $\forall x \forall y(Z(x) \vee Z(y) \rightarrow Z(x \cdot y))$.

(DG) $\forall x \forall y(\deg(x \cdot y) = \deg(x) + \deg(y))$.

(Z) $\forall g \exists! x(\deg(x) = g \wedge Z(x))$.

(S1) $\forall x \forall y(\deg(x) = \deg(y) \rightarrow \exists! z S(x, y, z))$.

(S2) $\forall x \forall y \forall z(S(x, y, z) \rightarrow \deg(x) = \deg(y) = \deg(z))$.

Using axioms (Z) and (S1) we introduce the following notation. For any g we write 0_g for the unique x such that $\deg(x) = g \wedge Z(x)$. If $\deg(x) = \deg(y) = g$, then we denote the unique z such that $S(x, y, z)$ by $x +_g y$. Furthermore, we set $A_g := \deg^{-1}(g)$. Note that $0_g \in A_g$, moreover by axiom (S2) we have that $+_g$ is a binary operation on A_g . Then the axioms of $\mathbf{T}_{gr}$ also include:

(AB) $(A_g, +_g, 0_g)$ is an abelian group for all g.

(DIS) The operation $\cdot$ is distributive over $+_g$ for all g.

We have not written all this axioms as first order sentences since it is straightforward to do it and that would have resulted in a long and meaningless list of $\mathcal{L}_{an}$-sentences. Let us just give an example here. We want to express that 0_g is the identity for $+_g$ in A_g for all g. This can be achieved via the following $\mathcal{L}_{an}$-sentence

$$\forall g \forall x \forall y \forall z(\deg(x) = g \wedge \deg(y) = g \wedge Z(y) \wedge S(x, y, z) \rightarrow z = x).$$

We will call *anneid* any model of $\mathbf{T}_{gr}$. By the axioms of $\mathbf{T}_{gr}$, the graded ring associated to any of its models will satisfy the additional properties (GR3) and (GR4). Conversely, if A is a Γ-graded ring satisfying (GR3) and (GR4), then $(H(A), \Gamma)$ is an anneid and A is the graded ring associated to it.

In order to study graded rings in more generality, one can relax the axioms (OAG) and (AGR) and restrict the language $\mathcal{L}_{an}$ further. For example, one might omit the relation symbol $<$ on the grading group sort language and instead of the axioms (OAG) use just the axioms of groups.

In the next chapter, we will only consider anneids of the form $(H(\mathrm{gr}_v(K)), vK)$ for some valued field (K, v) which are models of the above presented $\mathcal{L}_{an}$-theory $\mathbf{T}_{gr}$.

Chapter 5

Relative quantifier elimination

In this chapter we present some quantifier elimination results for valued fields, relative to valued hyperfields. One way of formalizing relative quantifier elimination is through many-sorted languages. We will adopt another approach and for this reason we give the following definition.

Definition 5.1. Let $\mathcal{L}$, $\mathcal{L}_i$ $(i \in I)$ be languages, for some index set I. Assume that for every $i \in I$ and for any $\mathcal{L}$-structure $\mathfrak{A}$ there is an $\mathcal{L}_i$-structure $\mathfrak{A}_i$ (called *i-structure* of $\mathfrak{A}$) and a surjective map

$$[\,\cdot\,]_i : A' \to A_i$$

from some $A' \subseteq A$ onto the universe A_i of $\mathfrak{A}_i$, such that

(SUB) if $\mathfrak{S}$ is an $\mathcal{L}$-substructure of $\mathfrak{A}$, then $\mathfrak{S}_i$ is an $\mathcal{L}_i$-substructure of $\mathfrak{A}_i$;

(TR) for all $i \in I$, all $\mathcal{L}_i$-formulae $\varphi = \varphi(x_1, \ldots, x_n)$ have a *translation.* That is, an $\mathcal{L}$-formula $\varphi_i = \varphi_i(x_1, \ldots, x_n)$ such that

$$(\mathfrak{A}_i, A_i) \models \varphi([a_1]_i, \ldots, [a_n]_i) \iff (\mathfrak{A}, A) \models \varphi_i(a_1, \ldots, a_n)$$

for every $\mathcal{L}$-structure $\mathfrak{A}$ and all $a_1, \ldots, a_n \in A'$.

Then we say that an $\mathcal{L}$-theory $\mathbf{T}$ is *substructure complete relative to the i-structures* $(i \in I)$ if for any two models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathbf{T}$ with a common substructure $\mathfrak{S}$, we have that the condition

$$\mathfrak{A}_i \equiv_{\mathfrak{S}_i} \mathfrak{B}_i \quad \text{for all } i \in I \tag{5.1}$$

implies $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$.

In Appendix A we relate relative substructure completeness to a syntactical notion of relative quantifier elimination. Roughly speaking, the above definition says that to check substructure completeness for **T** it suffices to check condition (5.1) for the i-structures associated to the models of **T** and their substructures. Recall that substructure completeness is equivalent to quantifier elimination (see Theorem 1.31).

This approach was used for instance in [26], where it is proved, e.g., that the theory of henselian valued fields with residue characteristic 0 is substructure complete relative to the amc-structure of level 0. In this case, $\mathcal{L}$ is $\mathcal{L}_{vf}$, $I = \{i_0\}$ is a singleton, $\mathcal{L}_{i_0} = \mathcal{L}_{amc}$ and the i_0-structure of a henselian valued field of residue characteristic 0 is its amc-structure of level 0. Note that in this case $\mathcal{L}_{i_0}$ is two-sorted. In this particular context, recalling notations from Section 4.2, we understand $[\,\cdot\,]_{i_0}$ to be given as the pair of surjective maps (π_0, π_0^*) and condition (TR) has to be interpreted accordingly: for all $a_1, \dots, a_n \in \mathcal{O}_v$ and all $b_1, \dots, b_m \in K^\times$,

$$K_0 \models \varphi(\pi_0 a_1, \dots, \pi_0 a_n, \pi_0^* b_1, \dots, \pi_0^* b_m) \iff (K, v) \models \varphi_{i_0}(a_1, \dots, a_n, b_1, \dots, b_m).$$

The first section contains our main theorem (Theorem 5.5). This result was achieved with the goal of rephrasing one of the main theorems of [26] in terms of valued hyperfields. With this result at hand, we will be able to obtain substructure completeness relative to the 0-valued hyperfield for the theory of henselian valued fields with residue characteristic 0 (Theorem 5.11) and substructure completeness relative to the $n \cdot vp$-valued hyperfields ($n \in \mathbb{N}$) for the theory of henselian valued fields of mixed characteristic $(0, p)$ (Corollary 5.32).

In Section 5.2 below, we will also discuss some connections between the language of valued hyperfields $\mathcal{L}_{vh}$ and the Denef-Pas language $\mathcal{L}_{DP}$ as well as establish a substructure completeness result relative to the graded ring structure for henselian valued fields of residue characteristic 0.

5.1 On the ultrapowers of $\mathcal{H}_\gamma(K)$ and K_γ

We need to start with two technical lemmas.

Lemma 5.2. *Fix a non-empty set S and an ultrafilter $\mathcal{U}$ on S. Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. Denote by (K^*, v^*) the ultrapower $(K, v)^S/\mathcal{U}$ and by $(\mathcal{H}_\gamma(K)^*, v_\gamma^*)$ the ultrapower $(\mathcal{H}_\gamma(K), v_\gamma)^S/\mathcal{U}$. Then*

$$(\mathcal{H}_\gamma(K)^*, v_\gamma^*) \simeq (\mathcal{H}_{\gamma^*}(K^*), v_{\gamma^*}^*)$$

where $\gamma^* := [(\gamma)]_{\mathcal{U}} \in vK^S/\mathcal{U}$.

Proof. We define

$$\sigma : \mathcal{H}_{\gamma^*}(K^*) \to \mathcal{H}_\gamma(K)^*$$
$$[[(a^{(s)})_s]_\mathcal{U}]_{\gamma^*} \mapsto [([a^{(s)}]_\gamma)_s]_\mathcal{U}$$

and show that it is an isomorphism of valued hyperfields.

In order to show that σ is well-defined assume that $[[(a^{(s)})_s]_\mathcal{U}]_{\gamma^*} = [[(b^{(s)})_s]_\mathcal{U}]_{\gamma^*}$. That is, there exists $[(t^{(s)})_s]_\mathcal{U} \in [(1)]_\mathcal{U} + \mathcal{M}^{\gamma^*}$ such that

$$[(a^{(s)})_s]_\mathcal{U} = [(t^{(s)})_s]_\mathcal{U}[(b^{(s)})_s]_\mathcal{U} = [(t^{(s)}b^{(s)})_s]_\mathcal{U}.$$

This is equivalent to

$$\{s \in S \mid a^{(s)} = t^{(s)}b^{(s)}\} \in \mathcal{U}.$$

By the definition of γ^* and since $[(t^{(s)})_s]_\mathcal{U} \in [(1)]_\mathcal{U} + \mathcal{M}^{\gamma^*}$ we have that

$$\{s \in S \mid v(t^{(s)} - 1) > \gamma\} \in \mathcal{U}$$

Therefore, the intersection of these two sets is an element of $\mathcal{U}$ by property (F2) of filters. However, this intersection equals to

$$\{s \in S \mid [a^{(s)}]_\gamma = [b^{(s)}]_\gamma\}.$$

Thus, $[([a^{(s)}]_\gamma)_s]_\mathcal{U} = [([b^{(s)}]_\gamma)_s]_\mathcal{U}$ as we wished to show.

We now show that σ is injective. Assume that $[([a^{(s)}]_\gamma)_s]_\mathcal{U} = [([b^{(s)}]_\gamma)_s]_\mathcal{U}$. That is,

$$D := \{s \in S \mid [a^{(s)}]_\gamma = [b^{(s)}]_\gamma\} \in \mathcal{U}.$$

Hence, for every $s \in D$ there exists $t^{(s)} \in 1 + \mathcal{M}^\gamma$ such that $a^{(s)} = t^{(s)}b^{(s)}$. Thus,

$$\{s \in S \mid a^{(s)} = t^{(s)}b^{(s)}\} = D \in \mathcal{U}.$$

For $s \in S \setminus D$ set $t^{(s)}$ to be any element of K. If we show that $[(t^{(s)})_s]_\mathcal{U} \in [(1)]_\mathcal{U} + \mathcal{M}^{\gamma^*}$, then we will obtain that $[[(a^{(s)})_s]_\mathcal{U}]_{\gamma^*} = [[(b^{(s)})_s]_\mathcal{U}]_{\gamma^*}$ as desired. We have

$$[(t^{(s)})_s]_\mathcal{U} - [(1)]_\mathcal{U} = [(t^{(s)} - 1)]_\mathcal{U}$$

and

$$\{s \in S \mid v(t^{(s)} - 1) > \gamma\} \supseteq D \in \mathcal{U}.$$

Hence, $v^*([(t^{(s)})_s]_\mathcal{U} - [(1)]_\mathcal{U}) > \gamma^*$ by (F3), proving what we wanted.

We now show that σ is surjective. Let $[(x^{(s)})_s]_\mathcal{U}$ be an element of $\mathcal{H}_\gamma(K)^*$. For each $s \in S$ take $a^{(s)} \in x^{(s)}$. We obtain,

$$\{s \in S \mid [a^{(s)}]_\gamma = x^{(s)}\} = S \in \mathcal{U}$$

and therefore

$$\sigma[[(a^{(s)})_s]_{\mathcal{U}}]_{\gamma^*} = [([a^{(s)}]_\gamma)_s]_{\mathcal{U}} = [(x^{(s)})_s]_{\mathcal{U}}.$$

In order to show that σ is a strict homomorphism of hyperrings we recall Lemma 2.9 and compute

$$\begin{aligned}
&\sigma([[(a^{(s)})_s]_{\mathcal{U}}]_{\gamma^*} + [[(b^{(s)})_s]_{\mathcal{U}}]_{\gamma^*}) = \\
&= \{\sigma[[(a^{(s)})_s]_{\mathcal{U}}[(t^{(s)})_s]_{\mathcal{U}} + [(b^{(s)})_s]_{\mathcal{U}}[(u^{(s)})_s]_{\mathcal{U}}]_{\gamma^*} \mid [(t^{(s)})_s]_{\mathcal{U}}, [(u^{(s)})_s]_{\mathcal{U}} \in [(1)]_{\mathcal{U}} + \mathcal{M}^{\gamma^*}\} = \\
&= \{\sigma[[(a^{(s)}t^{(s)} + b^{(s)}u^{(s)})_s]_{\mathcal{U}}]_{\gamma^*} \mid [(t^{(s)})_s]_{\mathcal{U}}, [(u^{(s)})_s]_{\mathcal{U}} \in [(1)]_{\mathcal{U}} + \mathcal{M}^{\gamma^*}\} = \\
&= \{[([a^{(s)}t^{(s)} + b^{(s)}u^{(s)}]_\gamma)_s]_{\mathcal{U}} \mid [(t^{(s)})_s]_{\mathcal{U}}, [(u^{(s)})_s]_{\mathcal{U}} \in [(1)]_{\mathcal{U}} + \mathcal{M}^{\gamma^*}\} = \\
&= [([a^{(s)}]_\gamma)_s]_{\mathcal{U}} + [([b^{(s)}]_\gamma)_s]_{\mathcal{U}}.
\end{aligned}$$

To justify the last equality, note that

$$\{s \in S \mid v(t^{(s)} - 1) > \gamma\}, \{s \in S \mid v(u^{(s)} - 1) > \gamma\} \in \mathcal{U}$$

and therefore

$$\{s \in S \mid [a^{(s)}t^{(s)} + b^{(s)}u^{(s)}]_\gamma \in [a^{(s)}]_\gamma + [b^{(s)}]_\gamma\} \in \mathcal{U}.$$

showing "$\subseteq$". For the other inclusion, observe that

$$[([c^{(s)}]_\gamma)_s]_{\mathcal{U}} \in [([a^{(s)}]_\gamma)_s]_{\mathcal{U}} + [([b^{(s)}]_\gamma)_s]_{\mathcal{U}}$$

if and only if

$$\{s \in S \mid [c^{(s)}]_\gamma \in [a^{(s)}]_\gamma + [b^{(s)}]_\gamma\} \in \mathcal{U}$$

meaning that

$$\{s \in S \mid c^{(s)} = a^{(s)}t^{(s)} + b^{(s)}u^{(s)} \text{ for some } t^{(s)}, u^{(s)} \in 1 + \mathcal{M}^\gamma\} \in \mathcal{U}$$

and thus

$$[([c^{(s)}]_\gamma)_s]_{\mathcal{U}} = [([a^{(s)}t^{(s)} + b^{(s)}u^{(s)}]_\gamma)_s]_{\mathcal{U}}$$

with $[(t^{(s)})_s]_{\mathcal{U}}, [(u^{(s)})_s]_{\mathcal{U}} \in [(1)]_{\mathcal{U}} + \mathcal{M}^{\gamma^*}$.

Regarding multiplication, we have

$$\begin{aligned}
\sigma([[(a^{(s)})_s]_{\mathcal{U}}]_{\gamma^*}[[(b^{(s)})_s]_{\mathcal{U}}]_{\gamma^*}) &= \sigma([[(a^{(s)}b^{(s)})_s]_{\mathcal{U}}]_{\gamma^*} \\
&= [([a^{(s)}b^{(s)}]_\gamma)_s]_{\mathcal{U}} \\
&= [([a^{(s)}]_\gamma)_s]_{\mathcal{U}}[([b^{(s)}]_\gamma)_s]_{\mathcal{U}}.
\end{aligned}$$

Clearly, σ satisfies (HH1) and therefore we have shown that it is a strict homomorphism of hyperrings.

It remains to show that

$$v^*_{\gamma^*}[[(a^{(s)})_s]_\mathcal{U}]_{\gamma^*} \geq [(0)]_\mathcal{U} \quad \Longleftrightarrow \quad v^*_\gamma[([a^{(s)}]_\gamma)_s]_\mathcal{U} \geq [(0)]_\mathcal{U}$$

Assume that $v^*_{\gamma^*}[[(a^{(s)})_s]_\mathcal{U}]_{\gamma^*} \geq [(0)]_\mathcal{U}$. This means that

$$v^*[(a^{(s)})_s]_\mathcal{U} \geq [(0)]_\mathcal{U}. \tag{5.2}$$

By definition on v^*, this is equivalent to

$$\{s \in S \mid va^{(s)} \geq 0\} \in \mathcal{U}$$

and, by definition of v_γ,

$$\{s \in S \mid va^{(s)} \geq 0\} = \{s \in S \mid v_\gamma[a^{(s)}]_\gamma \geq 0\}.$$

Therefore, (5.2) holds if and only if

$$\{s \in S \mid v_\gamma[a^{(s)}]_\gamma \geq 0\} \in \mathcal{U}.$$

This means that

$$v^*_\gamma[([a^{(s)}]_\gamma)_s]_\mathcal{U} \geq [(0)]_\mathcal{U}$$

and the proof is now complete. □

We have shown that the ultrapower of $(\mathcal{H}_\gamma(K), v_\gamma)$ with respect to an ultrafilter $\mathcal{U}$ over a non-empty set S is isomorphic to the γ^*-valued hyperfield of the ultrapower of (K, v) with respect to the same ultrafilter $\mathcal{U}$ over the same non-empty set S.

We now would like to show a similar result in the context of amc-structures which are here considered as $\mathcal{L}_{amc}$-structures.

Remark 5.3. We have not treated ultrapowers in many-sorted languages. However, the ultraproduct construction carries over to many-sorted logic without any specific trouble. Let us discuss here how is the ultrapower of an amc-structure defined. Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. Further, take a non-empty set S and an ultrafilter $\mathcal{U}$ on S. Then $K^S_\gamma/\mathcal{U}$ is the $\mathcal{L}_{amc}$-structure which has $(\mathcal{O}^\gamma)^S/\mathcal{U}$ as universe of the first sort (here the ultrapower is taken in the (single-sorted) language of rings) and $(G^\gamma)^S/\mathcal{U}$ as universe of the second sort (here the ultrapower is taken in the (single-sorted) language of groups). The relation between these two sorts, denoted by Θ^*_γ , is defined as follows. For $[(x^{(s)})_s]_\mathcal{U} \in (\mathcal{O}^\gamma)^S/\mathcal{U}$ and $[(y^{(s)})_s]_\mathcal{U} \in (G^\gamma)^S/\mathcal{U}$, one sets $\Theta^*_\gamma([(x^{(s)})_s]_\mathcal{U}, [(y^{(s)})_s]_\mathcal{U})$ to hold if and only if

$$\{s \in S \mid \Theta_\gamma(x^{(s)}, y^{(s)})\} \in \mathcal{U}.$$

Lemma 5.4. *Fix a non-empty set S and an ultrafilter $\mathcal{U}$ on S. Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. Denote by (K^*, v^*) the ultrapower $(K, v)^S/\mathcal{U}$ and by $(K_\gamma)^* = ((\mathcal{O}^\gamma)^*, (G^\gamma)^*, \Theta^*_\gamma)$ the ultrapower $K_\gamma^S/\mathcal{U}$. Then*

$$(K_\gamma)^* \simeq (K^*)_{\gamma^*}$$

where $\gamma^* := [(\gamma)]_\mathcal{U} \in vK^S/\mathcal{U}$.

Proof. By the previous lemma we immediately obtain an isomorphism (of groups) $\sigma_g : G^{\gamma^*} \to (G^\gamma)^*$.

We further define

$$\sigma_r : \mathcal{O}^{\gamma^*} \to (\mathcal{O}^\gamma)^*$$
$$[(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*} \mapsto [(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}$$

In order to see that σ_r is well-defined assume that

$$[(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*} = [(b^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}.$$

That is,

$$v^*([(a^{(s)})_s]_\mathcal{U} - [(b^{(s)})_s]_\mathcal{U}) > \gamma^*$$

so that

$$\{s \in S \mid v(a^{(s)} - b^{(s)}) > \gamma\} \in \mathcal{U}.$$

Thus,

$$\{s \in S \mid a^{(s)} + \mathcal{M}^\gamma = b^{(s)} + \mathcal{M}^\gamma\} = \{s \in S \mid v(a^{(s)} - b^{(s)}) > \gamma\} \in \mathcal{U}$$

which shows that $[(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U} = [(b^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}$.

We now show that σ_r is injective. Assume that

$$[(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U} = [(b^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}.$$

That is,

$$\{s \in S \mid a^{(s)} + \mathcal{M}^\gamma = b^{(s)} + \mathcal{M}^\gamma\} \in \mathcal{U}.$$

However, as we have already pointed out above

$$\{s \in S \mid v(a^{(s)} - b^{(s)}) > \gamma\} = \{s \in S \mid a^{(s)} + \mathcal{M}^\gamma = b^{(s)} + \mathcal{M}^\gamma\} \in \mathcal{U}.$$

This means that

$$v^*([(a^{(s)})_s]_\mathcal{U} - [(b^{(s)})_s]_\mathcal{U}) > \gamma^*$$

so that $[(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*} = [(b^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}$.

To show that σ_r is surjective we take an arbitrary element $[(\bar{a}^{(s)})_s]_\mathcal{U}$ and pick for each $s \in S$ a representative $a^{(s)} \in K$ of $\bar{a}^{(s)}$. Then

$$\{s \in S \mid a^{(s)} + \mathcal{M}^\gamma = \bar{a}^{(s)}\} = S \in \mathcal{U}.$$

Thus,

$$[(\bar{a}^{(s)})_s]_\mathcal{U} = [(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U} = \sigma_r([(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}).$$

In order to prove that σ_r is an homomorphism of rings, we compute

$$\begin{aligned}\sigma_r([(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*} + [(b^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}) &= \sigma_r([(a^{(s)} + b^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*})\\ &= [(a^{(s)} + b^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}\\ &= [(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U} + [(b^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}\\ &= \sigma_r([(a^{(s)})_s]_\mathcal{U}) + \sigma_r([(b^{(s)})_s]_\mathcal{U}).\end{aligned}$$

As for the product, we obtain

$$\begin{aligned}\sigma_r(([(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*})([(b^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*})) &= \sigma_r([(a^{(s)}b^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*})\\ &= [(a^{(s)}b^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}\\ &= [(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}[(b^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}\\ &= \sigma_r([(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*})\sigma_r([(b^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}).\end{aligned}$$

It remains to show that

$$\Theta_{\gamma^*}([(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}, [[(b^{(s)})_s]_\mathcal{U}]_{\gamma^*}) \Longleftrightarrow \Theta^*_\gamma([(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}, [([b^{(s)}]_\gamma)_s]_\mathcal{U})$$

for all $[(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*} \in \mathcal{O}^{\gamma^*}$ and $[[(b^{(s)})_s]_\mathcal{U}]_{\gamma^*} \in G^{\gamma^*}$.

Assume that $\Theta_{\gamma^*}([(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}, [[(b^{(s)})_s]_\mathcal{U}]_{\gamma^*})$ holds in $(K^*)_{\gamma^*}$. This means that there exists $[(c^{(s)})]_\mathcal{U} \in K^*$ such that

(*i*) $v^*[(c^{(s)})_s]_\mathcal{U} \geq [(0)]_\mathcal{U}$;

(*ii*) $v^*([(c^{(s)})_s]_\mathcal{U} - [(a^{(s)})]_\mathcal{U}) > \gamma^*$;

(*iii*) $v^*([(\frac{c^{(s)}}{b^{(s)}} - 1)_s]_\mathcal{U}) > \gamma^*$.

Now (*i*) is equivalent to

$$A := \{s \in S \mid vc^{(s)} \geq 0\} \in \mathcal{U}.$$

Further, (*ii*) is equivalent to

$$\{s \in S \mid v(c^{(s)} - a^{(s)}) > \gamma\} \in \mathcal{U}.$$

Since

$$\{s \in S \mid c^{(s)} + \mathcal{M}^\gamma = a^{(s)} + \mathcal{M}^\gamma\} = \{s \in S \mid v(c^{(s)} - a^{(s)}) > \gamma\},$$

that means that

$$B := \{s \in S \mid c^{(s)} + \mathcal{M}^\gamma = a^{(s)} + \mathcal{M}^\gamma\} \in \mathcal{U}$$

Finally, (iii) is equivalent to

$$\left\{s \in S \mid v\left(\frac{c^{(s)}}{b^{(s)}} - 1\right) > \gamma\right\} \in \mathcal{U}$$

that is,

$$C := \{s \in S \mid [c^{(s)}]_\gamma = [b^{(s)}]_\gamma\} \in \mathcal{U}.$$

Therefore,

$$\{s \in S \mid \Theta_\gamma(a^{(s)} + \mathcal{M}^\gamma, [b^{(s)}]_\gamma)\} = A \cap B \cap C \in \mathcal{U}$$

by $(F2)$, which means that

$$\Theta^*_\gamma([(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}, [([b^{(s)}]_\gamma)_s]_\mathcal{U})$$

holds in $(K_\gamma)^*$.

Assume now that $\Theta^*_\gamma([(a^{(s)} + \mathcal{M}^\gamma)_s]_\mathcal{U}, [([b^{(s)}]_\gamma)_s]_\mathcal{U})$ holds in $(K_\gamma)^*$. Namely,

$$D := \{s \in S \mid \Theta_\gamma(a^{(s)} + \mathcal{M}^\gamma, [b^{(s)}]_\gamma)\} \in \mathcal{U}.$$

Thus, for $s \in D$ we have $c^{(s)} \in K$ such that $vc^{(s)} \geq 0$, $v(c^{(s)} - a^{(s)}) > \gamma$ and $v(\frac{c^{(s)}}{b^{(s)}} - 1) > \gamma$. For $s \in S \setminus D$ let $c^{(s)}$ be any element of K. We have that the sets A, B and C defined above, contain D, thus by (F3) we obtain $A, B, C \in \mathcal{U}$ and as we have seen this is equivalent to $\Theta_{\gamma^*}([(a^{(s)})_s]_\mathcal{U} + \mathcal{M}^{\gamma^*}, [[(b^{(s)})_s]_\mathcal{U}]_{\gamma^*})$. The proof is now complete. □

We are now ready to state and prove the main theorem of this section.

Theorem 5.5. *Let* (K, v) *be a common valued subfield of* (L, w) *and* (F, u) *and take* $\gamma \in vK_{\geq 0}$ *. Then*

$$L_\gamma \equiv_{K_\gamma} F_\gamma \iff (\mathcal{H}_\gamma(L), w_\gamma) \equiv_{(\mathcal{H}_\gamma(K), v_\gamma)} (\mathcal{H}_\gamma(F), u_\gamma).$$

Proof. By Theorem 1.46 $L_\gamma \equiv_{K_\gamma} F_\gamma$ is equivalent to

$$(L_\gamma, K_\gamma)^* \simeq (F_\gamma, K_\gamma)^*$$

where $(L_\gamma, K_\gamma)^*$ is an ultrapower of (L_γ, K_γ) and $(F_\gamma, K_\gamma)^*$ is the ultrapower of (F_γ, K_γ) with respect to the same ultrafilter over the same set. Recall from Remark 1.44 that these amc-structures are nothing but the ultrapowers $(L_\gamma)^*$ and $(F_\gamma)^*$ where we interpret the parameters from K_γ as classes of constant sequences in $(K_\gamma)^*$. Thus, an isomorphism

$$(L_\gamma, K_\gamma)^* \simeq (F_\gamma, K_\gamma)^*$$

is just an isomorphism of $(L_\gamma)^*$ onto $(F_\gamma)^*$ which fixes the elements of $(K_\gamma)^*$ which are classes of constant sequences.

Applying the previous lemma we obtain an isomorphism $(L^*)_{\gamma^*} \simeq (F^*)_{\gamma^*}$. We now show that this isomorphism fixes the elements of $(K^*)_{\gamma^*}$ corresponding to constant sequences in K. Indeed, take $[(a^{(s)})_s]_{\mathcal{U}} + \mathcal{M}^{\gamma^*} \in \mathcal{O}_{v^*}^{\gamma^*}$ with $(a^{(s)})_s$ constant. Applying the isomorphism described in the proof of Lemma 5.4 yields $[(a^{(s)} + \mathcal{M}^\gamma)_s]_{\mathcal{U}} \in (\mathcal{O}_v^\gamma)^*$. Since $(a^{(s)})_s$ is constant, by assumption the latter is fixed by the isomorphism of $(L_\gamma)^*$ onto $(F_\gamma)^*$ and applying the isomorphism described in the proof of Lemma 5.4 we are sent back to $[(a^{(s)})_s]_{\mathcal{U}} + \mathcal{M}^{\gamma^*} \in \mathcal{O}_{v^*}^{\gamma^*}$ which is thus fixed. Similarly, if we take an element $[[(a^{(s)})_s]_{\mathcal{U}}]_{\gamma^*}$ of the group sort of $(K^*)_{\gamma^*}$ coming from a constant sequence $(a^{(s)})_s$, then applying the isomorphism described in the proof of Lemma 5.2 yields $[([a^{(s)}]_\gamma)_s]_{\mathcal{U}}$ in $(K_\gamma)^*$. Now, since $(a^{(s)})_s$ is constant, by assumption this is fixed by the isomorphism of $(L_\gamma)^*$ onto $(F_\gamma)^*$ and applying the isomorphism described in the proof of Lemma 5.2 yields again $[[(a^{(s)})_s]_{\mathcal{U}}]_{\gamma^*}$. This shows that the isomorphism $(L^*)_{\gamma^*} \simeq (F^*)_{\gamma^*}$ fixes the elements of $(K^*)_{\gamma^*}$ coming from constant sequences in K.

By Theorem 4.9 we have an isomorphism

$$(\mathcal{H}_{\gamma^*}(L^*), w^*_{\gamma^*}) \simeq (\mathcal{H}_{\gamma^*}(F^*), u^*_{\gamma^*})$$

which fixes the elements of $\mathcal{H}_{\gamma^*}(K^*)$ coming from constant sequences in K (see also Remark 4.11). Now, by Lemma 5.2 we obtain that the ultrapowers $(\mathcal{H}_\gamma(L)^*, w^*_\gamma)$ and $(\mathcal{H}_\gamma(F)^*, u^*_\gamma)$ are isomorphic via an isomorphism that fixes the elements of $\mathcal{H}_\gamma(K)^*$ corresponding to constant sequences. This can be justified with the following argument. Take an element $[a]_\gamma$ of $\mathcal{H}_\gamma(K)$. Since $[([a]_\gamma)]_{\mathcal{U}}$ is the image of $[[(a)]_{\mathcal{U}}]_{\gamma^*}$ under the isomorphism described in the proof of Lemma 5.2, applying the inverse of that isomorphism must yield $[[(a)]_{\mathcal{U}}]_{\gamma^*}$ which is an element of $\mathcal{H}_{\gamma^*}(K^*)$ coming from a constant sequence in K. Therefore, it is fixed by the isomorphism of $(\mathcal{H}_{\gamma^*}(L^*), w^*_{\gamma^*})$ onto $(\mathcal{H}_{\gamma^*}(F^*), u^*_{\gamma^*})$ and via the isomorphism described in the proof of Lemma 5.2 is sent back to $[([a]_\gamma)]_{\mathcal{U}}$. This shows that the element of $\mathcal{H}_\gamma(K)^*$ corresponding to the constant sequence $([a]_\gamma)_s$ is fixed as contended.

Applying Theorem 1.46 we now obtain

$$(\mathcal{H}_\gamma(L), w_\gamma) \equiv_{(\mathcal{H}_\gamma(K), v_\gamma)} (\mathcal{H}_\gamma(F), u_\gamma).$$

The proof of the converse implication goes along the same lines. We just have to use Theorem 4.10 in place of Theorem 4.9. For assume that

$$(\mathcal{H}_\gamma(L), w_\gamma) \equiv_{(\mathcal{H}_\gamma(K), v_\gamma)} (\mathcal{H}_\gamma(F), u_\gamma).$$

By Theorem 1.46, this is equivalent to $\mathcal{H}_\gamma(L)^* \simeq \mathcal{H}_\gamma(F)^*$ via an isomorphism which fixes the elements of $\mathcal{H}_\gamma(K)^*$ which are classes of constant sequences. By Lemma 5.2 this yields an isomorphism $\mathcal{H}_{\gamma^*}(L^*) \simeq \mathcal{H}_{\gamma^*}(F^*)$. This isomorphism fixes the elements of $\mathcal{H}_{\gamma^*}(K^*)$ coming from constant sequences in K, as it can be seen with a similar argument as above.

Theorem 4.10 now yields an isomorphism $(L^*)_{\gamma^*} \simeq (F^*)_{\gamma^*}$, which fixes the elements of $(K^*)_{\gamma^*}$ coming from constant sequences in K (cf. Remark 4.11).

Thus, by Lemma 5.4 we obtain an isomorphism $(L_\gamma)^* \simeq (F_\gamma)^*$ and it can be shown similarly as above that this fixes the elements of $(K_\gamma)^*$ which are classes of constant sequences. Therefore, the ultrapowers $(L_\gamma, K_\gamma)^*$ and $(F_\gamma, K_\gamma)^*$ of the canonical expansions of L_γ and F_γ to $\mathcal{L}_{amc}(K_\gamma)$ are isomorphic and Theorem 1.46 yields $L_\gamma \equiv_{K_\gamma} F_\gamma$. □

5.2 Henselian valued fields of residue characteristic 0

Let us start by observing that the $\mathcal{L}_{vf}$-theory of henselian valued fields with residue characteristic 0 is not substructure complete relative to the value group and the residue field.

Example 5.6. Let t be transcendental over $\mathbb{Q}$. Consider $L := \mathbb{Q}((t))$ and $F := \mathbb{Q}((t\sqrt{2}))$. Set $K := \mathbb{Q}((t^2))$ and let w be the t-adic valuation on L and u be the $t\sqrt{2}$-adic valuation on F. Then both (L, w) and (F, u) are henselian valued fields of residue characteristic 0. Moreover, K is clearly a subfield of L. Endow K with the valuation $v := w{\restriction}K$. Since,

$$t^2 = \frac{1}{2}(t\sqrt{2})^2 \in F,$$

we obtain that K is also a subfield of F and it follows that $(u{\restriction}K)(x) = vx$ for all $x \in K$. This shows that (K, v) is a substructure of (L, w) as well as of (F, u) in the language $\mathcal{L}_{vf}$ of valued fields. We have that $Lw \equiv_{Kv} Fu$ holds since $Lw = Fu = Kv = \mathbb{Q}$. Further, $wL \equiv_{vK} uF$ holds as well since $wL = uF = \mathbb{Z}$ and $vK = 2\mathbb{Z}$. Nevertheless, $(L, w) \equiv_{(K,v)} (F, u)$ does not hold since in L we have that $t^2 \in K$ is a square, while the same is not true in F because $t \notin F$.

Definition 5.7. Let $\mathcal{L}$ be a first order language and $\mathfrak{A}$ be an $\mathcal{L}$-structure. We say that $D \subseteq A$ is *definable* in $\mathcal{L}$ if there is an $\mathcal{L}$-formula $\varphi_D(x)$ such that

$$D = \{a \in A \mid \mathfrak{A} \models \varphi_D(a)\}.$$

Proposition 5.8. *The valuation hyperring $\mathcal{O}_{v_\gamma}$ of $(\mathcal{H}_\gamma(K), v_\gamma)$ is definable in $\mathcal{L}_{hf}$ extended with a constant symbol c whose interpretation is an element of value γ.*

Proof. We will abuse notation and write c also for the interpretation of the constant symbol c in the structure $\mathcal{H}_\gamma(K)$. We claim that

$$\mathcal{O}_{v_\gamma} = \{a \in \mathcal{H}_\gamma(K) \mid \neg r_+(ca^{-1}, 1, 1)\}.$$

To show this, observe that by Corollary 4.6

$$\neg r_+(ca^{-1}, 1, 1) \iff v_\gamma(ca^{-1}) \leq \gamma.$$

Using the properties of valuations and since $v_\gamma c = \gamma$, we see that the latter is equivalent to $v_\gamma c - v_\gamma a \leq v_\gamma c$ and then to $v_\gamma a \geq 0$. □

Since we always have 1 as a constant symbol in $\mathcal{L}_{hf}$ and the value of its interpretation must be 0 (cf. Lemma 2.36), the above proposition yields:

Corollary 5.9. *The valuation hyperring of $\mathcal{H}_0(K)$ is definable in $\mathcal{L}_{hf}$.*

To give a first application of Theorem 5.5 we recall the following result which is [26, Corollary 2.2]. Here amc-structures are considered as $\mathcal{L}_{amc}$-structures and valued fields as $\mathcal{L}_{vf}$-structures.

Theorem 5.10 ([26])**.** *The theory of henselian valued fields with residue characteristic 0 is substructure complete relative to the amc-structure of level 0.*

That is, if (L, w) and (F, u) are henselian valued fields with residue characteristic 0 and (K, v) is a common valued subfield of (L, w) and (F, u), then $L_0 \equiv_{K_0} F_0$ implies $(L, w) \equiv_{(K,v)} (F, u)$.

By Theorem 5.5, $L_0 \equiv_{K_0} F_0$ is equivalent to $(\mathcal{H}_0(L), w_0) \equiv_{(\mathcal{H}_0(K),v_0)} (\mathcal{H}_0(F), u_0)$. By Corollary 5.9 the latter is equivalent to $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$. Hence, we immediately derive the following result.

Theorem 5.11. *The theory of henselian valued fields with residue characteristic 0 is substructure complete relative to the 0-valued hyperfield.*

That is, if (L, w) and (F, u) are henselian valued fields with residue characteristic 0 and (K, v) is a common valued subfield of (L, w) and (F, u), then $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ (in $\mathcal{L}_{hf}$) implies $(L, w) \equiv_{(K,v)} (F, u)$.

In what follows we will apply this result to deduce other relative substructure completeness results for valued fields.

The following lemma is a refined version of Proposition 4.17. It will be used to establish a relation between $\mathcal{L}_{hf}$ and $\mathcal{L}_{DP}$.

Lemma 5.12. *Let (K, v) be a valued field which admits an angular component map α. Then for all $x, y \in K$ such that $vx \leq vy$ we have that*

$$[x]_0 + [y]_0 = \{[z]_0 \in \mathcal{H}_0(K) \mid \oplus_{ac}(x, y, z)\}$$

where $\oplus_{ac}(x, y, z)$ is the $\mathcal{L}_{DP}$-formula of type $(\mathbf{RF}, \mathbf{VG})$ given by

$$(\alpha(z) - \alpha(y) = \alpha(x) \wedge vx = vy = vz) \vee (\alpha(z) = \alpha(x) \wedge vx = vz < vy) \\ \vee (-\alpha(y) = \alpha(x) \wedge vx = vy < vz).$$

Proof. Assume first that $[z]_0 \in [x]_0 + [y]_0$. If $x = 0$, then by assumption $y = 0$, whence $z = 0$ and so $vx = vy = vz$ and $\alpha(z) - \alpha(y) = \alpha(x)$ follows hence $\oplus_{ac}(x, y, z)$ holds. Therefore, we can assume that $x \in K^\times$. Since $[z]_0 \in [x]_0 + [y]_0$, we have that $vz \geq vx$ by (V3). By Proposition 4.17 we obtain $\alpha(z - y) = \alpha(x)$ and $v(z - y) = vx$. We distinguish three cases:

- If $vx = vy = vz$, then $v(z - y) = vz$, hence by Lemma 4.18 we obtain that $\alpha(z) - \alpha(y) = \alpha(x)$.
- If $vx = vz < vy$, then using Lemma 4.16 we deduce $\alpha(x) = \alpha(z - y) = \alpha(z)$.
- If $vx = vy < vz$, then by Lemma 4.16 and Lemma 4.19 we obtain that $\alpha(x) = \alpha(z - y) = \alpha(-y) = -\alpha(y)$.

Observe that this case distinction is exhaustive by Part 4. of Lemma 2.36. We have shown that $\oplus_{ac}(x, y, z)$ holds when $[z]_0 \in [x]_0 + [y]_0$.

Conversely, assume that $\oplus_{ac}(x, y, z)$ holds. If $x = 0$, then it follows that $y = z = 0$, so $[z]_0 \in [x]_0 + [y]_0$. Hence, we may assume that $x \in K^\times$. We will prove that $[z]_0 \in [x]_0 + [y]_0$ by showing that $\alpha(z-y) = \alpha(x)$ and $v(z-y) = vx$ and then use Proposition 4.17.

We again distinguish three cases:

- If $vx = vy = vz$ and $\alpha(z) - \alpha(y) = \alpha(x)$, then suppose that $v(z - y) > vy$. We obtain that $\alpha(z) = \alpha(z - y + y) = \alpha(y)$ by Lemma 4.16. Therefore, $\alpha(x) = \alpha(z) - \alpha(y) = 0$ and $x = 0$ by (AC1). However, we have already treated this case. Hence, $v(z - y) = vy = vz$ and therefore by Lemma 4.18 we also have that $\alpha(x) = \alpha(z) - \alpha(y) = \alpha(z - y)$ as we wished to show.
- If $vx = vz < vy$ and $\alpha(z) = \alpha(x)$, then $v(z - y) = vz = vx$ and by Lemma 4.16 we have that $\alpha(z - y) = \alpha(z) = \alpha(x)$.
- If If $vx = vy < vz$ and $-\alpha(y) = \alpha(x)$, then $v(z - y) = v(-y) = vx$ and $\alpha(z - y) = \alpha(-y) = -\alpha(y) = \alpha(x)$ by Lemma 4.16 and Lemma 4.19. □

Lemma 5.13. *Consider an ac-valued field* $\mathfrak{K} = (K, Kv, vK, v, \alpha)$ *and let* $\varphi(x_1, \ldots, x_n)$ *be an* $\mathcal{L}_{hf}$*-formula. There exists an* $\mathcal{L}_{DP}$*-formula* $\varphi_{DP}(x_1, \ldots, x_{2n})$ *of type* $(\mathbf{RF}, \mathbf{VG})$ *such that for all* $a_1, \ldots, a_n \in K$

$$\mathcal{H}_0(K) \models \varphi([a_1]_0, \ldots, [a_n]_0) \iff \mathfrak{K} \models \varphi_{DP}(\alpha(a_1), \ldots, \alpha(a_n), va_1, \ldots, va_n).$$

Proof. We construct φ_{DP} by induction on the complecity of φ. Recall that $\mathcal{H}_0(K)$ is isomorphic as a hyperfield via $[x]_0 \mapsto (\alpha(x), vx)$ to $Kv \oplus (vK \cup \{\infty\})$ by Theorem 4.22. Hence by Proposition 1.21 we may identify these hyperfields.

Now, if φ is an equality, say, $[a]_0 = [b]_0$, then φ_{DP} is $\alpha(a) = \alpha(b) \wedge va = vb$.

If φ is $r_+([a]_0, [b]_0, [c]_0)$, then by the previous lemma, we can take φ_{DP} to be

$$\oplus_{ac}(a, b, c) \vee \oplus_{ac}(b, a, c).$$

If φ is $\psi \wedge \theta$, then we set φ_{DP} to be $\psi_{DP} \wedge \theta_{DP}$ and if φ is $\neg\psi$, then we set φ_{DP} to be $\neg\psi_{DP}$. If φ is $\forall(r, \gamma)\psi$, where (r, γ) denotes a variable for $Kv \oplus (vK \cup \{\infty\})$, then φ_{DP} is $\forall r \forall \gamma \psi_{DP}$ where $\forall r$ is a quantifier over the **RF** sort and $\forall \gamma$ is a quantifier over the **VG** sort. That φ_{DP} is of type $(\mathbf{RF}, \mathbf{VG})$ follows by construction. □

By Pas' Theorem (cf. Theorem 1.47) we know that in presence of an angular component map in the language, the theory of henselian valued fields of residue characteristic 0 admits quantifier elimination relative to the value group and the residue field. If we look back at Example 5.6, we indeed see that $\mathbb{Q}((t))$ and $\mathbb{Q}((t\sqrt{2}))$ induce different angular component maps on $\mathbb{Q}((t^2))$. The following result, which we derive from Theorem 5.11, may be seen as a version of Pas' theorem.

Theorem 5.14. *If* $\mathfrak{L} = (L, wL, Lw, w, \alpha_L)$ *and* $\mathfrak{F} = (F, uF, Fu, u, \alpha_F)$ *are henselian ac-valued fields with residue characteristic* 0 *and* $\mathfrak{K} = (K, vK, Kv, v, \alpha)$ *is a common ac-valued subfield of* $\mathfrak{L}$ *and* $\mathfrak{F}$*, then* $Lw \equiv_{Kv} Fu$ *(in* $\mathcal{L}_f$*) and* $wL \equiv_{vK} uF$ *(in* $\mathcal{L}_{og} \cup \{\infty\}$*) imply* $(L, w) \equiv_{(K,v)} (F, u)$ *(in* $\mathcal{L}_{vf}$*).*

Proof. We show that $wL \equiv_{vK} uF$ and $Lw \equiv_{Kv} Fu$ imply $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ and then conclude using Theorem 5.11.

Let $\varphi = \varphi([a_1]_0, \ldots, [a_n]_0)$ be an $\mathcal{L}_{hf}(\mathcal{H}_0(K))$-sentence which holds in $\mathcal{H}_0(L)$. This happens if and only if $\varphi_{DP} = \varphi_{DP}(\alpha(a_1), \ldots, \alpha(a_n), va_1, \ldots, va_n)$ holds in $\mathfrak{L}$, by the previous lemma. Note that by construction φ_{DP} will have parameters from Kv and vK. Since φ_{DP} is of type $(\mathbf{RF}, \mathbf{VG})$ we find sentences $\varphi_R = \varphi_R(\alpha(a_1), \ldots, \alpha(a_n))$ and $\varphi_G = \varphi_G(va_1, \ldots, va_n)$ in the language of fields and in the language of ordered abelian groups extended with ∞, respectively, such that

$$\mathfrak{L} \models \varphi_{DP} \iff Lw \models \varphi_R \text{ and } wL \models \varphi_G\ .$$

Now our assumption says that the right hand side is equivalent to

$$Fu \models \varphi_R \text{ and } uF \models \varphi_G \ .$$

which is equivalent to $\mathfrak{F} \models \varphi_{DP}$. This last assertion is equivalent to φ holding in $\mathcal{H}_0(F)$ by the previous lemma, so we have proved what we wanted. □

We will now show that the theory of henselian valued fields with residue characteristic 0 is substructure complete relative to the anneid structure and the graded ring structure.

We start with the anneid structure. Let (K, v) be a valued field. We will abuse notation and write $\mathrm{gr}_v(K)$ also to refer to the $\mathcal{L}_{an}$-structure $(H(\mathrm{gr}_v(K)), vK)$ given by the set of homogeneous elements of $\mathrm{gr}_v(K)$ and its grading group vK, to which $\mathrm{gr}_v(K)$ is associated.

We will make use of the following observation.

Lemma 5.15. *Let (K, v) be a valued field and $\varphi(x_1, \ldots, x_n)$ an $\mathcal{L}_{hf}$-formula. There exists an $\mathcal{L}_{an}$-formula $\varphi_{an}(x_1, \ldots, x_n)$ such that for all $a_1, \ldots, a_n \in K$*

$$\mathcal{H}_0(K) \models \varphi([a_1]_0, \ldots, [a_n]_0) \quad \Longleftrightarrow \quad \mathrm{gr}_v(K) \models \varphi_{an}(\mathrm{in}_v(a_1), \ldots, \mathrm{in}_v(a_n)).$$

Proof. We construct φ_{an} by induction on the complexity of φ. If φ is an equality $[a]_0 = [b]_0$, then we can take φ_{an} to be $\mathrm{in}_v(a) = \mathrm{in}_v(b)$. This works by Lemma 4.26. If φ is $r_+([a]_0, [b]_0, [c]_0)$, then we set φ_{an} to be the disjuction of the following four $\mathcal{L}_{an}$-sentences, where we write x for $\mathrm{in}_v(a)$, y for $\mathrm{in}_v(b)$ and z for $\mathrm{in}_v(c)$.

- $\deg(x) < \deg(y) \wedge z = x$;
- $\deg(y) < \deg(x) \wedge z = y$;
- $\deg(x) = \deg(y) =: \gamma \wedge \neg Z(x +_\gamma y) \wedge z = x +_\gamma y$;
- $\deg(x) = \deg(y) =: \gamma \wedge Z(x +_\gamma y) \wedge (z = x +_\gamma y \vee \deg(z) > \gamma)$.

According to the notation introduced in listing the axioms of $\mathbf{T}_{gr}$, when $\deg(x) = \deg(y) =: \gamma$,we have denoted by $x +_\gamma y$ the (unique) element such that $S(x, y, x +_\gamma y)$ holds. The above mentioned disjuction expresses exactly that $z \in x \boxplus y$ where $\boxplus$ is the hyperoperation of the wedge sum (4.10) of the abelian groups $\mathcal{P}^\gamma/\mathcal{M}^\gamma$. We have shown in Section 4.4 that this is equivalent to $r_+([a]_0, [b]_0, [c]_0)$ as we want.

If φ is $\psi \wedge \theta$, then we set φ_{an} to be $\psi_{an} \wedge \theta_{an}$. If φ is $\neg\psi$, then we set φ_{an} to be $\neg\psi_{an}$. Finally, if φ is $\forall x \psi$, then we set φ_{an} to be $\forall x \psi_{an}$, where $\forall x$ is a quantifier over the sort $\mathbf{H}$ of $\mathcal{L}_{an}$. Then that the assertion of the lemma holds is straightforward to verify. □

Remark 5.16. Let (K, v) be a valued field and (L, w) an extension of it. Then vK embeds in wL as usual and we have an injective map $H(\mathrm{gr}_v(K)) \to H(\mathrm{gr}_w(L))$ defined via the assignment $\mathrm{in}_v(x) \mapsto \mathrm{in}_w(x)$, $x \in K$. That this map is injective follows from Lemma 4.26 and the fact that ϕ, the map introduced before Corollary 3.31, is injective.

Let us show that this injective map is an $\mathcal{L}_{zs}$-embedding. It clearly respects $\cdot$, since $\mathrm{in}_v(x) \cdot \mathrm{in}_v(y) = \mathrm{in}_v(xy)$ for all $x, y \in K$. Moreover, for $x \in K$ we have that $Z(\mathrm{in}_v(x))$ holds in $\mathrm{gr}_v(K)$ if and only if $x = 0$ holds in K. This might seem strange, but recall that when we defined graded rings, we identified all the zeros of $\mathcal{P}^\gamma/\mathcal{M}^\gamma$ with the zero of the graded ring which is by convention $\mathrm{in}_v(0)$. Therefore, if $\mathrm{in}_v(x)$ is a zero of some $\mathcal{P}^\gamma/\mathcal{M}^\gamma$, then it must be $\mathrm{in}_v(0)$, i.e., x must be 0. The converse implication is clear. Clearly, $x = 0$ holds in K if and only if $x = 0$ holds in L and this is equivalent to $Z(\mathrm{in}_w(x))$ holding in $\mathrm{gr}_w(L)$.

For $x, y \in K^\times$ and $z \in K$ we have that $S(\mathrm{in}_v(x), \mathrm{in}_v(y), \mathrm{in}_v(z))$ holds in $\mathrm{gr}_v(K)$ if and only if $vx = vy$ and $v(z - (x + y)) > vx$ holds in K which is equivalent to $wx = wy$ and $w(z - (x + y)) > wx$ holding in L (since $x, y, z \in K$) and the latter means that $S(\mathrm{in}_w(x), \mathrm{in}_w(y), \mathrm{in}_w(z))$. If $x = 0$ and $y, z \in K$, then $S(\mathrm{in}_v(x), \mathrm{in}_v(y), \mathrm{in}_v(z))$ is equivalent to $\mathrm{in}_v(y) = \mathrm{in}_v(z)$. This (by injectivity) is equivalent to $\mathrm{in}_w(y) = \mathrm{in}_w(z)$ which means $S(\mathrm{in}_w(x), \mathrm{in}_w(y), \mathrm{in}_w(z))$ since $x = 0$. The case $y = 0$ and $x, z \in K$ is analogous. Putting these three cases together, we have shown that $S(\mathrm{in}_v(x), \mathrm{in}_v(y), \mathrm{in}_v(z))$ holds in $\mathrm{gr}_v(K)$ if and only if $S(\mathrm{in}_w(x), \mathrm{in}_w(y), \mathrm{in}_w(z))$ holds in $\mathrm{gr}_w(L)$. Thus, we indeed have an $\mathcal{L}_{zs}$-embedding.

Since for all $x \in K$ we have $vx = wx$, the $\mathcal{L}_{zs}$-embedding $H(\mathrm{gr}_v(K)) \hookrightarrow H(\mathrm{gr}_w(L))$ and the $\mathcal{L}_{og}$-embedding $vK \hookrightarrow wL$ defined above, preserve the degree map, i.e., the following diagram

$$\begin{array}{ccc} H(\mathrm{gr}_v(K)) & \xrightarrow{\deg_K} & vK \\ \big\downarrow & & \big\downarrow \\ H(\mathrm{gr}_w(L)) & \xrightarrow[\deg_L]{} & wL \end{array}$$

commutes. This shows that these two embeddings form an $\mathcal{L}_{an}$-embedding.

In what follows we will identify $\mathrm{gr}_v(K)$ with its image under this $\mathcal{L}_{an}$-embedding thereby regarding $\mathrm{gr}_v(K)$ as an $\mathcal{L}_{an}$-substructure of $\mathrm{gr}_w(L)$.

In the following substructure completeness result we will have $\mathcal{L} = \mathcal{L}_{vf}$, I will be a singleton $\{i_0\}$ and $\mathcal{L}_{i_0} = \mathcal{L}_{an}$. Given a valued field (K, v) we have surjective maps $\mathrm{in}_v : K \to H(\mathrm{gr}_v(K))$ and $v : K^\times \to vK$. In analogy to what it is done for amc-structures, $[\ \cdot\ ]_{i_0}$ will be understood as the pair (in_v, v). The adapted condition (TR) will then require that for all $\mathcal{L}_{an}$-formula φ there is a $\mathcal{L}_{vf}$-formula φ_{i_0} such that for all $a_1, \ldots, a_n \in K$ and all $b_1, \ldots, b_m \in K^\times$

$$\mathrm{gr}_v(K) \models \varphi(\mathrm{in}_v(a_1), \ldots, \mathrm{in}_v(a_n), vb_1, \ldots, vb_m) \iff (K, v) \models \varphi_{i_0}(a_1, \ldots, a_n, b_1, \ldots, b_m).$$

We will now show that this is indeed true. Since in $\mathcal{L}_{an}$ there are no relation symbols of type $(\mathbf{H}, \mathbf{G})$, the given $\mathcal{L}_{an}$-formula φ can be written as $\psi \wedge \theta$ where ψ is an $\mathcal{L}_{zs}$-formula and θ is an $\mathcal{L}_{og}$-formula. Hence, φ_{i_0} can be constructed by translating ψ and θ. Since we know that a translation θ_g of θ in $\mathcal{L}_{vf}$ exists, it remains to find a translation ψ_{zs} of the $\mathcal{L}_{zs}$-formula ψ. This is the content of the next lemma.

Lemma 5.17. *Let (K, v) be a valued field and $\varphi(x_1, \dots, x_n)$ an $\mathcal{L}_{zs}$-formula. There exists an $\mathcal{L}_{vf}$-formula $\varphi_{zs}(x_1, \dots, x_n)$ such that for all $a_1, \dots, a_n \in K$,*

$$\mathrm{gr}_v(K) \vDash \varphi(\mathrm{in}_v(a_1), \dots, \mathrm{in}_v(a_n)) \iff (K, v) \vDash \varphi_{zs}(a_1, \dots, a_n).$$

Proof. We construct φ_{zs} by induction on the complexity of φ. If φ is an equality, say $\mathrm{in}_v(a) = \mathrm{in}_v(b)$, then, translating φ is the same as translating $[a]_0 = [b]_0$ by Lemma 4.26. This has already been done in Corollary 3.4.

If φ is $Z(\mathrm{in}_v(a))$, then φ_{zs} is equivalent to $a = 0$ as we have already discussed in Remark 5.16. In the same remark, distinguishing several cases, we have found φ_{zs} also when φ is $S(\mathrm{in}_v(a), \mathrm{in}_v(b), \mathrm{in}_v(c))$: if $a, b \in K^\times$ and $c \in K$ we can set φ_{zs} to be the obvious translation in $\mathcal{L}_{vf}$ of $va = vb \wedge v(c - (a+b)) > va$. On the other hand, if $a = 0$ or $b = 0$, then φ is just an equality and we have already treated this case above. This concludes the base step of the induction.

If φ is $\psi \wedge \theta$, then we set φ_{zs} to be $\psi_{zs} \wedge \theta_{zs}$ and if φ is $\neg\psi$, then we set φ_{zs} to be $\neg\psi_{zs}$. Finally, if φ is $\forall x \psi$, then φ_{zs} is $\forall x \psi_{zs}$. Then that the assertion of the lemma holds is straightforward to verify. □

Theorem 5.18. *The theory of henselian valued fields of residue characteristic 0 is substructure complete relative to the anneid structure.*

That is, if (L, w) and (F, u) are henselian valued fields with residue characteristic 0 and (K, v) is a common valued subfield, then $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ (in $\mathcal{L}_{an}$) implies $(L, w) \equiv_{(K,v)} (F, u)$ (in $\mathcal{L}_{vf}$).

Proof. We are going to show that $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ in $\mathcal{L}_{an}$ implies $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ and then conclude using Theorem 5.11.

Let $\varphi = \varphi([a_1]_0, \dots, [a_n]_0)$ be an $\mathcal{L}_{hf}$-sentence with parameters from $\mathcal{H}_0(K)$, which holds in $\mathcal{H}_0(L)$. By the Lemma 5.15, this happens if and only if

$$\mathrm{gr}_w(L) \vDash \varphi_{an}(\mathrm{in}_v(a_1), \dots, \mathrm{in}_v(a_n)). \tag{5.3}$$

Note that φ_{an} is an $\mathcal{L}_{an}$-sentence with parameters from $\mathrm{gr}_v(K)$, therefore by our assumption (5.3) is equivalent to

$$\mathrm{gr}_u(F) \vDash \varphi_{an}(\mathrm{in}_v(a_1), \dots, \mathrm{in}_v(a_n)).$$

Again by Lemma 5.15, this means that φ holds in $\mathcal{H}_0(F)$, as we wished to show. □

Some readers might consider the above result not satisfactory as in the assumption $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ (in $\mathcal{L}_{an}$) we are not actually talking about the graded rings themselves, but about the anneids to which the graded rings are associated. In fact, we can still ask what one can conclude when the graded rings themselves are elementarily equivalent (in some suitable language). For this, one can add a sort to the language $\mathcal{L}_{an}$ for the associated graded ring and put on it, for instance, the language of rings. In this way, one obtains a three-sorted language $\mathcal{L}$ and the anneids we have considered above will be just reducts of the $\mathcal{L}$-structures. Clearly, if two $\mathcal{L}$-structures are elementarily equivalent, then their $\mathcal{L}_{an}$-reducts are and the above result may be applied.

However, there is also another way. For consider the graded rings as $\mathcal{L}_{gr}$-structures. Recall that $\mathcal{L}_{gr}$ is the language of rings extended with a unary function symbol g to be interpreted as the initial form function (cf. Definition 4.30). We will prove the following result.

Theorem 5.19. *The theory of henselian valued fields of residue characteristic* 0 *is substructure complete relative to the graded ring structure.*

That is, if (L, w) *and* (F, u) *are henselian valued fields with residue characteristic* 0 *and* (K, v) *is a common valued subfield, then* $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ *(in* $\mathcal{L}_{gr}$*) implies* $(L, w) \equiv_{(K,v)} (F, u)$ *(in* $\mathcal{L}_{vf}$*).*

Now the condition $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ (in $\mathcal{L}_{gr}$) is indeed a condition on the graded rings themselves.

For the next lemma, we define $H(x)$ to mean $\exists y : g(y) = x$. Thus, the interpretation of H will be the image of the initial form function g, i.e., the set of homogeneous elements of the graded ring (with ordered grading group) under consideration.

Lemma 5.20. *Let* (K, v) *be a valued field and* $\varphi(x_1, \ldots, x_n)$ *an* $\mathcal{L}_{hf}$*-formula. There exists an* $\mathcal{L}_{gr}$*-formula* $\varphi_{gr}(x_1, \ldots, x_n)$ *such that for all* $a_1, \ldots, a_n \in K$*,*

$$\mathcal{H}_0(K) \models \varphi([a_1]_0, \ldots, [a_n]_0) \iff \mathrm{gr}_v(K) \models \varphi_{gr}(\mathrm{in}_v(a_1), \ldots, \mathrm{in}_v(a_n)).$$

Proof. We construct φ_{gr} by induction on the complexity of φ. If φ is $[a]_0 = [b]_0$, then φ_{gr} is $\mathrm{in}_v(a) = \mathrm{in}_v(b)$. This works by Lemma 4.26.

If φ is $r_+([a]_0, [b]_0, [c]_0)$, then we set φ_{gr} to be

$$g(\mathrm{in}_v(c) \ominus \mathrm{in}_v(b)) = \mathrm{in}_v(a) \vee g(\mathrm{in}_v(c) \ominus \mathrm{in}_v(a)) = \mathrm{in}_v(b).$$

Recall here that we use the symbol $\ominus$ to denote the minus sign of $\mathrm{gr}_v(K)$. This works by Proposition 4.33.

If φ is $\psi \wedge \theta$, then φ_{gr} is $\psi_{gr} \wedge \theta_{gr}$ and if φ is $\neg\psi$, then φ_{gr} is $\neg\psi_{gr}$. Finally, if φ is $\forall x \psi$, then we set φ_{gr} to be $\forall x(H(x) \to \psi_{gr})$. Then that the assertion of the lemma holds is straightforward to verify. □

In the substructure completeness result stated in Theorem 5.19, we have that $\mathcal{L} = \mathcal{L}_{vf}$, $I = \{i_0\}$ is a singleton and $\mathcal{L}_{i_0} = \mathcal{L}_{gr}$. Before we go on with the proof, let us observe that there is a formal problem here. Indeed, we do not have a canonical choice for a surjective function $[\,\cdot\,]_{i_0} : K \to \mathrm{gr}_v(K)$ as required by Definition 5.1. However, we can still work with the surjective function $\mathrm{in}_v : K \to H(\mathrm{gr}_v(K))$.

We will now clarify how to interpret conditions (SUB) and (TR) of Definition 5.1 in this setting. Let us start with (TR). Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}_{gr}$-formula. Combining Lemma 5.20 and Corollary 3.4 we obtain that there is an $\mathcal{L}_{vf}$-formula φ' such that for all $a_1, \ldots, a_n \in K$,

$$\mathrm{gr}_v(K) \models \varphi(\mathrm{in}_v(a_1), \ldots, \mathrm{in}_v(a_n)) \iff (K, v) \models \varphi'(a_1, \ldots, a_n). \tag{5.4}$$

Now, let $b_1, \ldots, b_n \in \mathrm{gr}_v(K)$. For $1 \leq j \leq n$, we write

$$b_j = \mathrm{in}_v(a_{1,j}) \oplus \ldots \oplus \mathrm{in}_v(a_{m_j,j})$$

for some $m_j \in \mathbb{N}$ and $a_{k,j} \in K$, $1 \leq k \leq m_j \in \mathbb{N}$. We can then consider $\varphi(b_1, \ldots, b_n)$ as a sentence with $\mathrm{in}_v(a_{k,j})$ as parameters. From this point of view, it is possible to translate φ into φ' using (5.4). In doing so, we will have that

$$\begin{aligned} \mathrm{gr}_v(K) \models \varphi(b_1, \ldots, b_n) &\iff \mathrm{gr}_v(K) \models \varphi(\mathrm{in}_v(a_{1,1}), \ldots, \mathrm{in}_v(a_{m_n,n})) \\ &\iff (K, v) \models \varphi'(a_{1,1}, \ldots, a_{m_n,n}). \end{aligned}$$

The $\mathcal{L}_{vf}$-formula φ' thus obtained is then considered as the translation φ_{i_0} of φ.

Regarding condition (SUB), let (K, v) be a valued subfield of a valued field (L, w). We have to explain how to $\mathcal{L}_{gr}$-embed $\mathrm{gr}_v(K)$ into $\mathrm{gr}_w(L)$. Clearly, this can be done by means of the assignment $\mathrm{in}_v(x) \mapsto \mathrm{in}_w(x)$ for $x \in K$. It is also straightforward to verify that the induced injective map $\mathrm{gr}_v(K) \hookrightarrow \mathrm{gr}_w(L)$ respects the initial form function and is therefore an $\mathcal{L}_{gr}$-embedding. As before, when considering $\mathrm{gr}_v(K)$ as a substructure of $\mathrm{gr}_w(L)$ we are identifying $\mathrm{gr}_v(K)$ with its image in $\mathrm{gr}_w(L)$ under this embedding.

We are now ready to prove Theorem 5.19.

Proof of Theorem 5.19. We are going to show that $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ in $\mathcal{L}_{gr}$ implies $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ and then conclude using Theorem 5.11.

Let $\varphi = \varphi([a_1]_0, \ldots, [a_n]_0)$ be an $\mathcal{L}_{hf}$-sentence with parameters from $\mathcal{H}_0(K)$, which holds in $\mathcal{H}_0(L)$. By the Lemma 5.20, this happens if and only if

$$\mathrm{gr}_w(L) \models \varphi_{gr}(\mathrm{in}_v(a_1), \ldots, \mathrm{in}_v(a_n)). \tag{5.5}$$

Note that φ_{gr} is an $\mathcal{L}_{gr}$-sentence with parameters from $\mathrm{gr}_v(K)$, therefore by our assumption (5.5) is equivalent to

$$\mathrm{gr}_u(F) \models \varphi_{gr}(\mathrm{in}_v(a_1), \ldots, \mathrm{in}_v(a_n)).$$

Again by Lemma 5.20, this means that φ holds in $\mathcal{H}_0(F)$, as we wished to show. □

It should be at this point clear that $(L,w) \equiv_{(K,v)} (F,u)$ in $\mathcal{L}_{vf}$ implies that $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ in $\mathcal{L}_{hf}$ and $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ in $\mathcal{L}_{an}$ as well as in $\mathcal{L}_{gr}$. This follows by condition (TR) of Definition 5.1 which we have verified in all of these settings. Hence, we deduce from Theorem 5.11, Theorem 5.18 and Theorem 5.19 the following result.

Corollary 5.21. *Let (L,w) and (F,u) be henselian valued fields with residue characteristic 0 and (K,v) a common valued subfield. Then the following assertions are equivalent:*

1. $(L,w) \equiv_{(K,v)} (F,u)$ *in* $\mathcal{L}_{vf}$,
2. $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ *in* $\mathcal{L}_{hf}$,
3. $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ *in* $\mathcal{L}_{an}$,
4. $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ *in* $\mathcal{L}_{gr}$.

5.3 Henselian valued fields of mixed characteristic

Let us start with the well-known fundamental inequality. Let (K,v) be a valued field and $L|K$ be a finite extension of fields. By [15, Theorem 3.2.9] there are finitely many extensions $v_1,\dots,v_m$ of v from K to L. Then the fundamental inequality tells us that

$$[L:K] \geq \sum_{i=1}^{m}[Lv_i : Kv](v_iL : vK). \tag{5.6}$$

See [15, Theorem 3.3.4].

Definition 5.22. A valued field (K,v) is called *defectless* if for any finite extension $L|K$ equality holds in the fundamental inequality (5.6).

We will need the following non-trivial result. We are not in a position to prove it in full generality, hence we will only give reference to general results from which it follows.

Proposition 5.23. *A valued field of residue characteristic 0 is defectless.*

Reference for a proof. Let (K,v) be a valued field of residue characteristic 0. If (K,v) is henselian, then the result follows from the Lemma of Ostrowski [45, Theorem 2, p. 153]. For the general case, we refer to the discussion after Lemma 45 in [8]. In there it is shown that

$$[L:K] = \sum_{i=1}^{m} d_i(L|K,v_i)[Lv_i, Kv](v_iL : vK),$$

where for each $i = 1,\ldots,m$, $d_i(L|K,v_i)$ is a power of the residue characteristic exponent, which by definition is 1 in case the residue characteristic is 0. From this, our proposition clearly follows. □

Let (L,v) be a valued field and K a subfield of L. Then we denote the valuation $v\restriction K$ on K again by v and in this sense we will speak about extensions of valued fields $(L|K,v)$.

Definition 5.24. An extension of valued fields $(L|K,v)$ will be called *pre-tame* if the following holds:

(PT1) the residue field extension $Lv|Kv$ is separable,

(PT2) the order of every torsion element of vL/vK is not divisible by $\mathrm{char}\, Kv$.

For the definition of (not necessarily algebraic) separable extensions of fields we refer to [31]. Note that conditions (PT1) and (PT2) are always satisfied if the residue characteristic is 0.

Definition 5.25. A subgroup Δ of an ordered abelian group Γ is called *convex* if for all $\delta_1, \delta_2 \in \Delta$ and for all $\gamma \in \Gamma$ one has that $\delta_1 \leq \gamma \leq \delta_2$ implies $\gamma \in \Delta$.

If Δ is a convex subgroup of an ordered abelian group Γ, then the quotient group Γ/Δ can be ordered in such a way that the canonical epimorphism $\Gamma \to \Gamma/\Delta$ is order-preserving [15, Section 2.1].

Definition 5.26. Let v and w be valuations on a field K. Then w is a *coarsening* of v if $\mathcal{O}_v \subseteq \mathcal{O}_w$.

Let (K,v) be a valued field and fix a convex subgroup Δ of vK. There is a unique coarsening v_Δ of the valuation v whose value group is isomorphic to vK/Δ and whose residue field Kv_Δ carries a valuation $\bar{v}_\Delta$ with value group isomorphic to Δ such that v is the composition of v_Δ and $\bar{v}_\Delta$. The valuation v_Δ is defined as:

$$v_\Delta(x) := vx + \Delta.$$

For a nonzero $xv_\Delta \in Kv_\Delta$ one then sets

$$\bar{v}_\Delta(xv_\Delta) := vx.$$

Note that since xv_Δ is nonzero in Kv_Δ, the value of x under v_Δ is zero, i.e., $vx \in \Delta$. For more details we refer the reader to e.g. [15, Section 2.3]. If (K, v) is henselian, then (K, v_Δ) is henselian [15, Corollary 4.1.4].

Consider an arbitrary extension $(L|K, v)$. There is a unique smallest convex subgroup of vL containing the convex subgroup Δ of vK; it is the *convex hull* of Δ:

$$\Delta' := \{\gamma \in vL \mid \delta_1 \leq \gamma \leq \delta_2 \text{ for some } \delta_1, \delta_2 \in \Delta\}.$$

Observe that since Δ is convex, $\Delta' \cap vK = \Delta$. This implies that we have an order-preserving embedding

$$\begin{aligned} vK/\Delta &\to vL/\Delta' \\ \gamma + \Delta &\mapsto \gamma + \Delta' \end{aligned}$$

Since we identify equivalent valuations, the coarsening of the valuation v of L which corresponds to Δ' is an extension of v_Δ from K to L and will again be denoted by the same symbol v_Δ .

Definition 5.27. An extension $(L|K, v)$ of valued fields is called *immediate* if the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto.

Lemma 5.28. *A valued field extension $(L|K, v)$ is immediate if and only if for every $x \in L \setminus K$ there exists $a \in K$ such that $v(x - a) > vx$.*

Proof. Assume that $(L|K, v)$ is immediate. We will identify vK with vL and Kv with Lv. Take $x \in L \setminus K$. In particular $x \neq 0$ and hence $vx \in vL = vK$. Let $d \in K$ be such that $vdx = 0$ and $dxv \in Lv = Kv$. Let $b \in K$ be such that $bv = dxv$. Then $v(dx - b) > 0$, so $v(x - bd^{-1}) > -vd = vx$, hence $a := bd^{-1}$ satisfies $v(x - a) > vx$.

Assume now that for all $x \in L \setminus K$ there is $a \in K$ such that $v(x - a) > vx$. Pick any $\gamma \in vL$ and write $\gamma = vx$ for some $x \in L$. From $v(x - a) > vx$ it follows that $vx = va \in vK$. Now let $\zeta \in Lv$ and $x \in L$ be such that $xv = \zeta$. From $v(x - a) > vx = 0$ it follows that $xv = av \in Kv$. □

Proposition 5.29. *An extension $(L|K, v)$ of valued fields is immediate if and only if the canonical embedding $\phi : \mathcal{H}_0(K) \hookrightarrow \mathcal{H}_0(L)$ is onto.*

Proof. Assume that $(L|K, v)$ is immediate. Take $x \in L$ and consider $[x]_0 \in \mathcal{H}_0(L)$. By Part 1) of Lemma 3.3

$$[x]_0 = \{y \in L \mid v(x - y) > vx\}.$$

Now, if $x \in K$, then $[x]_0 \in \operatorname{Im} \phi$ and there is nothing to show. If $x \in L \setminus K$, then by the previous lemma there is $a \in K$ such that $a \in [x]_0$ and therefore $[x]_0 = [a]_0 \in \operatorname{Im} \phi$.

Assume that ϕ is onto. This implies that for all $x \in L \setminus K$, there is $a \in K$ such that $[x]_0 = [a]_0$. By Part 1) of Lemma 3.3 this means that for all $x \in L \setminus K$ there exists $a \in K$ such that $v(x-a) > vx$. We may then conclude that $(L|K, v)$ is immediate by virtue of the previous lemma. □

In [26] Kuhlmann considers elementary classes $\mathcal{K}$ of valued fields (always assumed to be nontrivially valued) which have the following properties:

(IME) If $(L|K, v)$ and $(F|K, v)$ are immediate extensions and $(K, v), (L, v), (F, v) \in \mathcal{K}$, then $(L, v) \equiv_{(K,v)} (F, v)$.

(RAC) If $(L, v) \in \mathcal{K}$, the quotient vL/vK is a torsion group and the extension $Lv|Kv$ is algebraic, then the relative algebraic closure L' of K in L with the valuation $v\lceil L'$ (again denoted by v) is an element of $\mathcal{K}$ and $(L|L', v)$ is immediate.

Combining [26, Theorem 2.1] with our Theorem 5.5 we derive the following result.

Theorem 5.30. *Let $\mathcal{K}$ be an elementary class of valued fields which satisfies (IME) and (RAC). Further, let (K, v) be a common valued subfield of the henselian fields (L, v) and (F, v). Suppose that Δ is a convex subgroup of vK such that*

(i) $(L^*, v_\Delta), (F^*, v_\Delta) \in \mathcal{K}$ *for all elementary extensions* (L^*, v) *and* (F^*, v) *of* (L, v) *and* (F, v) *on which* v_Δ *is nontrivial,*

(ii) (K, v_Δ) *is a defectless field,*

(iii) (L, v_Δ) *and* (F, v_Δ) *are pre-tame extensions of* (K, v_Δ).

Then the following statements are equivalent:

1. $(L, v) \equiv_{(K,v)} (F, v)$,
2. $(\mathcal{H}_\gamma(L), v_\gamma) \equiv_{(\mathcal{H}_\gamma(K), v_\gamma)} (\mathcal{H}_\gamma(F), v_\gamma)$ *for every* $0 \leq \gamma \in \Delta$.

For the sake of completeness we give a proof of the following proposition.

Proposition 5.31. *The elementary class of henselian valued fields of residue characteristic* 0 *satisfies (IME) and (RAC).*

Proof. We first show (IME). We recall the Ax-Kochen-Ershov principle valid for extension $(L|K, v)$ of henselian valued fields of residue characteristic 0:

$$vK \preceq vL \text{ and } Kv \preceq Lv \implies (K, v) \preceq (L, v)$$

A proof can for instance be found in [21, Theorem 6.17 (2)].

Take immediate extensions $(L|K,v)$ and $(F|K,v)$ with $(K,v),(L,v),(F,v)$ henselian valued fields of residue characteristic 0. Since $vK \preceq vL$ and $Kv \preceq Lv$ are trivially satisfied in this case, one obtains that $(K,v) \preceq (L,v)$. Similarly, one has that $(K,v) \preceq (F,v)$. These mean that $(K,v) \equiv_{(K,v)} (L,v)$ and $(K,v) \equiv_{(K,v)} (F,v)$ by Lemma 1.23. Hence, $(L,v) \equiv_{(K,v)} (F,v)$ follows.

We now show (RAC). Let (L,v) be a henselian valued field of residue characteristic 0 and (K,v) be a valued subfield of (L,v). Assume in addition that vL/vK is a torsion group and that $Lv|Kv$ is an algebraic extension. We consider the relative algebraic closure

$$L' := \{x \in L \mid x \text{ is algebraic over } K\}$$

of K in L, with the valuation $v = v\restriction L'$. Then $L'v$ is a subfield of Lv and therefore has characteristic 0. We now show that (L',v) is henselian. Let f be a monic polynomial with coefficients in the valuation ring $\mathcal{O}'$ of (L',v) and suppose that $b \in \mathcal{O}'$ is such that $vf(b) > 0$ and $vf'(b) = 0$. Since (L,v) is henselian and $\mathcal{O}'$ is contained in the valuation ring $\mathcal{O}$ of L, we obtain that f has a root $a \in L$ such that $av = bv$. Since L' is relatively algebraically closed in L we may conclude that $a \in L'$, being a root of a polynomial with coefficients in L'. We have shown that L' is henselian.

Let $\zeta \in Lv$. Since the extension $Lv|Kv$ is algebraic, there is a polynomial $f(X)$ with coefficients in the valuation ring of (K,v) such that the residue polynomial fv has ζ as a root. Since Lv, and hence Kv, has characteristic 0, ζ is a simple root of fv. Therefore, by henselianity of L there is a root $a \in L$ of f such that $av = \zeta$. This shows that $\zeta \in L'v$, since L' is relatively algebraically closed in L and thus $a \in L'$ must hold.

We have shown that $Lv = L'v$. We now want to prove that $vL = vL'$. Let $\alpha \in vL$. Since vL/vK is a torsion group, there exists $n \in \mathbb{N}$ such that $n\alpha \in vK$. Let $a \in L$ be such that $va = \alpha$ and take $b := a^n$. Let $c \in K$ be such that $vb = vc$. Let $d \in L'$ be such that $dv = (bc^{-1})v \in Lv = L'v$. Then $bc^{-1}d^{-1}$ has residue $1v$. Consider the polynomial $X^n - 1$ over Kv. Since $\mathrm{char}(Kv) = 0$ we have that $1v$ is a simple root of $X^n - 1$ and therefore the polynomial over K given by $X^n - bc^{-1}d^{-1}$ has a root $t \in L$ such that $tv = 1v$, since L is henselian. In particular, $t \in L'$, $vt = 0$ and $t^n = bc^{-1}d^{-1}$. Therefore, $cdt^n = b = a^n$. Now let $s := at^{-1}$. Clearly, we have that $va = vs$. Since $s^n = cd \in L'$ we obtain that $s \in L'$ since L' is relatively algebraically closed in L. This shows that $vL = vL'$ and thus $(L|L',v)$ is immediate. □

Let (K,v) be a valued field and assume that $\mathrm{char}\, K = 0$. Denote by p the *characteristic exponent* of Kv. That is, $p = 1$ if $\mathrm{char}\, Kv = 0$ and $p = \mathrm{char}\, Kv$ otherwise. The *canonical decomposition* of the valuation v is defined as follows.

Let Δ be the smallest convex subgroup of vK containing the value vp; note that $\{n \cdot vp \mid n \in \mathbb{N}\}$ is cofinal in Δ. That is, for all $\delta \in \Delta$ there is $n \in \mathbb{N}$ such that $n \cdot vp \geq \delta$. We write $\dot{v}$ for v_Δ; this is called the *coarse valuation* assigned to v. We have that $\dot{v} = v$ if and only if $p = 1$ and that $\dot{v}$ is trivial if and only if $\Delta = vK$. A valued field of characteristic 0 is of mixed characteristic if and only if $\dot{v}$ is coarser than v, i.e., $\Delta \neq \{0\}$. The valuation ring $\dot{\mathcal{O}}$ of $(K, \dot{v})$ is characterized as the smallest overring of $\mathcal{O}_v$ in which p becomes a unit. Namely, $\dot{\mathcal{O}}$ is the localization of $\mathcal{O}_v$ with respect to the multiplicatively closed set $\{p^n \mid n \in \mathbb{N}\}$ (see [15, Section 2.3]). Consequently, the residue field $K\dot{v}$ is of characteristic 0.

Notation: For $n \in \mathbb{N}$ we write $\mathcal{H}_n(K)$ instead of $\mathcal{H}_{n \cdot vp}(K)$ and v_n instead of $v_{n \cdot vp}$.

If (K, v) is a henselian valued field, then $(K, \dot{v})$ is a henselian valued field of residue characteristic 0 (possibly trivially valued). By Proposition 5.23, $(K, \dot{v})$ is a defectless valued field. Moreover, as we have already noted above, every extension of $(K, \dot{v})$ will be pre-tame.

Hence, with Δ as above and $\mathcal{K}$ the class of all henselian valued fields of residue characteristic 0, from Theorem 5.30 we obtain the following result.

Corollary 5.32. *The theory of henselian valued fields of mixed characteristic $(0, p)$ is substructure complete relative to the $n \cdot vp$-valued hyperfields ($n \in \mathbb{N}$).*

That is, if (L, w) and (F, u) are henselian valued fields of characteristic 0 with residue characteristic $p > 0$ and (K, v) is a common valued subfield of (L, w) and (F, u), then the condition

$$\mathcal{H}_n(L) \equiv_{\mathcal{H}_n(K)} \mathcal{H}_n(F) \text{ for every } n \in \mathbb{N}$$

implies $(L, w) \equiv_{(K,v)} (F, u)$.

Proof. Since $\{n \cdot vp \mid n \in \mathbb{N}\}$ is cofinal in Δ it suffices to have

$$(\mathcal{H}_n(L), w_n) \equiv_{(\mathcal{H}_n(K), v_n)} (\mathcal{H}_n(F), u_n) \text{ for every } n \in \mathbb{N}$$

in order to conclude that

$$(\mathcal{H}_\gamma(L), w_\gamma) \equiv_{(\mathcal{H}_\gamma(K), v_\gamma)} (\mathcal{H}_\gamma(F), u_\gamma) \text{ for every } \gamma \in \Delta.$$

This holds because, as shown in [26], the same is true for the corresponding amc-structures and we may apply Theorem 5.5.

Now, if we show that

$$(\mathcal{H}_n(L), w_n) \equiv_{(\mathcal{H}_n(K), v_n)} (\mathcal{H}_n(F), u_n) \text{ for every } n \in \mathbb{N}$$

is equivalent to

$$\mathcal{H}_n(L) \equiv_{\mathcal{H}_n(K)} \mathcal{H}_n(F) \text{ for every } n \in \mathbb{N},$$

then the proof will be complete. To achieve this we will now prove that, for all $n \in \mathbb{N}$, the valuation hyperring of $\mathcal{H}_n(K)$ is definable in $\mathcal{L}_{hf}$. The same argument can be used to show that the valuation hyperrings of $\mathcal{H}_n(L)$ and $\mathcal{H}_n(F)$ are definable in $\mathcal{L}_{hf}$.

For $n = 0$ this has been shown in Corollary 5.9. Take $n \in \mathbb{N} \setminus \{0\}$. We claim that

$$[0]_n \notin \underbrace{[1]_n + \ldots + [1]_n}_{p \text{ times}} =: I(p).$$

Indeed, if that would be so, then by Part 5) of Lemma 3.3 there would be $b \in \mathcal{M}^n$ such that $0 = p + b$. Hence, $n \cdot vp < vb = v(-b) = vp$, a contradiction.

We now show that

$$v_n(I(p)) = \{vp\}.$$

For $p = 2$ this is true since $[0]_n \notin [1]_n + [1]_n$ and $[2]_n \in [1]_n + [1]_n$, so we may apply Proposition 3.19. For $p > 2$ this is because by definition (cf. (2.1)),

$$I(p) = \bigcup_{[x]_n \in I(p-1)} [x]_n + [1]_n \ .$$

For $[x]_n \in I(p-1)$ we find $b \in \mathcal{M}^n$ such that

$$x = \underbrace{1 + \ldots + 1}_{p-1 \text{ times}} + b$$

by Part 5) of Lemma 3.3. Therefore,

$$v(x+1) = \min\{vp, vb\} = vp.$$

Now since $[0]_n \notin I(p)$, by Proposition 3.19 we have that

$$v_n([x]_n + [1]_n) = \{v(x+1)\} = \{vp\}$$

for all $[x]_n \in I(p-1)$. This shows that $v_n(I(p)) = \{vp\}$.

Now, $I(p)$ is definable in $\mathcal{L}_{hf}$ by the sentence $\varphi(x)$ given by

$$\exists x_1 \ldots \exists x_{p-2}(r_+(1,1,x_1) \wedge r_+(x_1,1,x_2) \wedge r_+(x_2,1,x_3) \wedge \ldots \wedge r_+(x_{p-2},1,x)).$$

For $p = 2$ we set $\varphi(x)$ to be just $r_+(1,1,x)$.

By Proposition 5.8, $\mathcal{O}_{v_n}$ is definable in $\mathcal{L}_{hf}$ extended with a constant symbol c to be interpreted as an element of value $\gamma = n \cdot vp$. Since any element of the definable set $I(p)$ to the power n will have this value, we obtain that

$$\mathcal{O}_{v_n} = \{a \in \mathcal{H}_n(K) \mid \exists x : \varphi(x) \wedge \neg r_+(x^n a^{-1}, 1, 1)\}$$

is definable in $\mathcal{L}_{hf}$ as we wished to prove. □

We conclude this section giving an example which shows that in the mixed characteristic case, neither the 0-valued hyperfield nor the graded ring or the anneid structure are sufficient in order to obtain relative substructure completeness. We will indeed show that there are henselian valued fields (L, w) and (F, u) of mixed characteristic, with a common valued subfield (K, v), which are not elementarily equivalent over (K, v), but $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ in $\mathcal{L}_{hf}$ as well as $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ in $\mathcal{L}_{gr}$ and in $\mathcal{L}_{an}$.

In this example we will use notation and terminology for p-valued fields as in [44].

Example 5.33. We take $K := \mathbb{Q}_2((t))$ to be the field of formal Laurent series over the field of 2-adic numbers. The valuation v on K that we consider is the composition of the t-adic valuation v_t with the 2-adic valuation v_2. That is, for nonzero

$$x = \sum a_i t^i,$$

we set $vx := (v_t x, v_2 a_{v_t x})$ (cf. [44, Example 2.3]). Thus, $vK = \mathbb{Z} \times \mathbb{Z}$ ordered lexicografically, the coarse valuation $\dot{v}$ assigned to v is the t-adic valuation v_t and the valuation on the residue $K\dot{v} = \mathbb{Q}_2$ is the 2-adic valuation (which in this setting is called the *core valuation* assigned to v). Since v is the composition of henselian valuations, we have that (K, v) is henselian (see [15, Corollary 4.1.4]). Moreover, the residue field Kv coincides with the residue field $K\dot{v}$ with respect to the core valuation, that is $Kv = \mathbb{F}_2$. The valued field (K, v) is a 2-valued field of 2-rank 1 (it is known in this case that the 2-rank of K coincides with the 2-rank of the *core field* $K\dot{v}$, see [44, Section 2.2] for details).

We further consider the Puiseux series fields over $\mathbb{Q}_2$ in t and in $3t$. That is, we take

$$L := \bigcup_{n\in\mathbb{N}} \mathbb{Q}_2((t^{\frac{1}{n}})) \quad \text{and} \quad F := \bigcup_{n\in\mathbb{N}} \mathbb{Q}_2(((3t)^{\frac{1}{n}}))$$

with the valuations w and u, defined similarly as v: the valuation w is the composition of the t-adic valuation on L with the 2-adic valuation and u is the composition of the $3t$-adic valuation on F with the 2-adic valuation (cf. [44, Example 2.4]). Note that, $wL = uF = \mathbb{Q} \times \mathbb{Z}$ is a $\mathbb{Z}$-group. Moreover, (L, w) and (F, u) are henselian since w and u are compositions of henselian valuations. It follows that (L, w) and (F, u) are 2-adically closed (see [44, Theorem 3.1]). Their residue field still coincides with the residue field of $\mathbb{Q}_2$ with respect to the 2-adic valuation, i.e., $\mathbb{F}_2$ and their 2-rank is 1.

Since $3 \in \mathbb{Q}_2$ and it has value $(0, 0)$ with respect to w and u, it follows that (K, v) is a common valued subfield of (L, w) and (F, u). Our first claim is that (L, w) and (F, u) are not elementarily equivalent over (K, v). Indeed, $t \in K$ is a square in L but the same is not true in F. For suppose that $t^{\frac{1}{2}} \in F$, then we would obtain that $3^{\frac{1}{2}} = (3t)^{\frac{1}{2}} t^{-\frac{1}{2}} \in F$. Since $3^{\frac{1}{2}}$ is algebraic over $\mathbb{Q}_2$, it follows that $3^{\frac{1}{2}}$

belongs to the relative algebraic closure of $\mathbb{Q}_2$ in F. Now, since $\mathbb{Q}_2$ is 2-adically closed of the same 2-rank as F, with respect to the restriction of u to $\mathbb{Q}_2$, namely, with respect to the 2-adic valuation, it follows that $\mathbb{Q}_2$ is relatively algebraically closed in F and therefore $3^{\frac{1}{2}} \in \mathbb{Q}_2$. This is a contradiction, since 3 is not a square in $\mathbb{Q}_2$ (because it is not congruent to 1 modulo 8, see e.g. [47, Theorem 4, p. 18]). Thus, the $\mathcal{L}_{vf}(K)$ sentence $\exists x : x^2 = t$ holds in (L, w) but not in (F, u) and we have shown that (L, w) and (F, u) are not elementarily equivalent over (K, v).

Our second claim is that $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ in $\mathcal{L}_{gr}$ as well as in $\mathcal{L}_{an}$. This will automatically imply that $\mathcal{H}_0(L) \equiv_{\mathcal{H}_0(K)} \mathcal{H}_0(F)$ in $\mathcal{L}_{hf}$ too, since e.g. Lemma 5.20 is valid for any valued field (not just for those of residue characteristic 0) and we may proceed as in the proof of Theorem 5.19.

The valuations w and u admit canonical sections:

$$\begin{aligned} \varepsilon : wL &\to L^\times \\ \left(\frac{a}{b}, c\right) &\mapsto 2^c t^{\frac{a}{b}} \end{aligned}$$

and

$$\begin{aligned} \epsilon : uF &\to F^\times \\ \left(\frac{a}{b}, c\right) &\mapsto 2^c (3t)^{\frac{a}{b}} \end{aligned}$$

In the terminology of [2], these are choice functions such that, for all $\gamma, \gamma' \in \mathbb{Q} \times \mathbb{Z}$,

$$\left(\frac{\varepsilon(\gamma)\varepsilon(\gamma')}{\varepsilon(\gamma+\gamma')}\right) w = 1w \quad \text{and} \quad \left(\frac{\epsilon(\gamma)\epsilon(\gamma')}{\epsilon(\gamma+\gamma')}\right) u = 1u.$$

Therefore (see [2, Section 2]), the maps

$$\psi_L(\mathrm{in}_w(x)) := \left(\frac{x}{\varepsilon \circ w(x)}\right) w \cdot X^{wx} \qquad (x \in L^\times)$$

and

$$\psi_F(\mathrm{in}_u(y)) := \left(\frac{y}{\epsilon \circ u(y)}\right) u \cdot X^{uy} \qquad (y \in F^\times)$$

extend to isomorphisms

$$\mathrm{gr}_w(L) \simeq Lw[X^{wL}] \quad \text{and} \quad \mathrm{gr}_u(F) \simeq Fu[X^{uF}]$$

by setting

$$\psi_L(\mathrm{in}_w(x_1) \oplus \ldots \oplus \mathrm{in}_w(x_k)) := \sum_{l=1}^{k} \psi_L(\mathrm{in}_w(x_l)),$$

and similarly for ψ_F.

Since $Fu = Lw = \mathbb{F}_2$ and $wL = uF = \mathbb{Q} \times \mathbb{Z}$, we obtain that, as rings, $\mathrm{gr}_w(L) \simeq \mathrm{gr}_u(F)$ via $\psi_F^{-1} \circ \psi_L$. We now show that this isomorphism is over $\mathrm{gr}_v(K)$ and that it preserve the initial form functions (i.e., it is an $\mathcal{L}_{gr}$-isomorphism). This by Theorem 1.21 will imply our claim in $\mathcal{L}_{gr}$.

In order to achieve this we first observe that, by definition, for all $(\frac{a}{b}, c) \in \mathbb{Q}\times\mathbb{Z}$ we have that

$$\epsilon\left(\frac{a}{b}, c\right) = 3^{\frac{a}{b}}\varepsilon\left(\frac{a}{b}, c\right).$$

Therefore, if $x \in K^\times$, then $wx = vx = ux$ and

$$\left(\frac{x}{\varepsilon \circ w(x)}\right) w \cdot X^{wx} = \left(3^{\dot{v}x}\frac{x}{\epsilon \circ v(x)}\right) v \cdot X^{vx} = \left(\frac{x}{\epsilon \circ u(x)}\right) u \cdot X^{ux},$$

where we have used the fact that 3 (and hence $3^{\dot{v}x}$, since $\dot{v}x \in \mathbb{Z}$) is a 1-unit in K, i.e., has residue 1. From this it follows that the isomorphism $\psi_F^{-1} \circ \psi_L$ fixes the elements of $\mathrm{gr}_v(K)$.

The ring $R := \mathbb{F}_2[t^{\mathbb{Q}\times\mathbb{Z}}]$ is a $\mathbb{Q} \times \mathbb{Z}$-graded ring and we see, as observed in [2, Remark 2.6], that ψ_L and ψ_F preserve the degree maps, i.e., for all $x \in L^\times$ we have that $\deg_L(\mathrm{in}_w(x)) = wx = \deg_R(\psi_L(\mathrm{in}_w(x)))$ and for all $y \in F^\times$ we have that $\deg_F(\mathrm{in}_u(y)) = uy = \deg_R(\psi_F(\mathrm{in}_u(y)))$, where $\deg_R$ is the natural degree map of R. Hence, if $\mathrm{in}_w(x)$ is of minimal degree in the respresentation of some nonzero element of $\mathrm{gr}_w(L)$, then $\psi_F^{-1} \circ \psi_L(\mathrm{in}_w(x))$ is of minimal degree in the representation of the corresponding element of $\mathrm{gr}_u(F)$. This shows that $\psi_F^{-1} \circ \psi_L$ preserves the initial form functions and is thus an $\mathcal{L}_{gr}$-ismorphism as claimed.

Finally, we observe that when restricted to the set of homogeneous elements, the isomorphism $\psi_F^{-1} \circ \psi_L$ induces an $\mathcal{L}_{an}$-isomorphism: this is given by the just mentioned restriction together with the identity $wL = uF$. As we have shown above, this isomorphism fixes the elements of the anneid $(H(\mathrm{gr}_v(K)), vK)$. We conclude that $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ in $\mathcal{L}_{an}$ too.

Appendix A

On relative quantifier elimination

In this appendix we prove that relative substructure completeness is equivalent to a syntactical notion, to be understood in the spirit of (relative) quantifier elimination. We essentially follow the idea of the proof of [3, Theorem B]. The statement of the general result is as follows.

Theorem A.1. *In the situation described in Definition 5.1, we have that an $\mathcal{L}$-theory $\mathbf{T}$ is substructure complete relative to the i-structures ($i \in I$) if and only if for any $\mathcal{L}$-formula φ there exist $i_1, \ldots, i_n \in I$ with $n \in \mathbb{N}$ and $\lambda^j \in F_{\mathcal{L}_{i_j}}$ for $1 \leq j \leq n$ such that φ is $\mathbf{T}$-equivalent to a finite disjuction of $\mathcal{L}$-formulae of the form*

$$\theta \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j}$$

where θ is quantifier-free.

Remark A.2. As they are translations of the $\mathcal{L}_{i_j}$-formulae λ^j, one can think about the $\mathcal{L}$-formulae $(\lambda^j)_{i_j}$ as formulae which only express properties of the corresponding i-structures. In this sense, we may say that relative substructure completeness is equivalent to relative quantifier elimination.

For the convenience of the reader we state the following lemma which is an immediate consequence of (TR).

Lemma A.3. *In the situation described in Definition 5.1, let $\mathfrak{S}$ be a substructure of an $\mathcal{L}$-structure $\mathfrak{A}$. Let $\varphi = \varphi([a_1]_i, \ldots, [a_n]_i)$ be an $\mathcal{L}_i(S_i)$-sentence for some $i \in I$ and $a_1, \ldots, a_n \in S$. Let $\varphi_i(x_1, \ldots, x_n)$ be the translation of the $\mathcal{L}_i$-formula $\varphi(x_1, \ldots, x_n)$ and consider the $\mathcal{L}(S)$-sentence $\varphi_i = \varphi_i(a_1, \ldots, a_n)$. Then*

$$(\mathfrak{A}_i, S_i) \models \varphi \quad \Longleftrightarrow \quad (\mathfrak{A}, S) \models \varphi_i \ .$$

Proof of Theorem A.1. We begin by proving that relative substructure completeness implies relative quantifier elimination. Let $\varphi = \varphi(x_1, \ldots, x_m)$ be an $\mathcal{L}$-formula. Let $c_1, \ldots, c_m$ be constant symbols which are not in $\mathcal{C}$. We define $\mathbf{A}(\varphi)$ to be the set of $\mathcal{L}(c_1, \ldots, c_m)$-sentences of the form

$$\neg \left(\theta(c_1, \ldots, c_m) \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j}(c_1, \ldots, c_m) \right),$$

where $n \in \mathbb{N}$, $\theta(x_1, \ldots, x_m)$ is a quantifier-free $\mathcal{L}$-formula and $\lambda^j(x_1, \ldots, x_m)$ is an $\mathcal{L}_{i_j}$-formula for all $1 \leq j \leq n \in \mathbb{N}$, such that

$$\mathbf{T} \models \left(\theta \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j} \right) \to \varphi.$$

We claim that the set of $\mathcal{L}(c_1, \ldots, c_m)$-sentences $\mathbf{S} := \mathbf{T} \cup \{\varphi(c_1, \ldots, c_m)\} \cup \mathbf{A}(\varphi)$ is inconsistent (i.e., it has no model). Assuming the contrary, let us choose a model $(\mathfrak{A}, \{c_1, \ldots, c_m\})$ of $\mathbf{S}$. We let $\mathfrak{S}$ be the intersection of all substructures of $\mathfrak{A}$ which contain $c_1, \ldots, c_m$. There is a canonical way to endow $\mathfrak{S}$ with an $\mathcal{L}$-structure and obtain a substructure of $\mathfrak{A}$. Denote by $\mathbf{D}(\mathfrak{S})$ the diagram of $\mathfrak{S}$, i.e., the set of all atomic and negations of atomic $\mathcal{L}(S)$-sentences which hold in $(\mathfrak{S}, S)$. It is known that models $(\mathfrak{B}, S)$ of $\mathbf{D}(\mathfrak{S})$ correspond to extensions $\mathfrak{B}$ of $\mathfrak{S}$ (cf. [43, Lemma 2.4.2]). We further consider the set $\mathbf{I}(\mathfrak{A})$ of all $\mathcal{L}(S)$-sentences having the form λ_i for some $i \in I$ and some $\mathcal{L}_i(S_i)$-sentence λ such that $(\mathfrak{A}_i, S_i) \models \lambda$. The $\mathcal{L}(S)$-sentences λ_i are obtained from the $\mathcal{L}_i(S_i)$-sentences λ as described in the statement of Lemma A.3.

Take a model $(\mathfrak{B}, S)$ of $\mathbf{T}' := \mathbf{T} \cup \mathbf{D}(\mathfrak{S}) \cup \mathbf{I}(\mathfrak{A})$. We claim that

$$(\mathfrak{B}, S) \equiv (\mathfrak{A}, S).$$

Indeed, since $\mathbf{T}$ is by assumption substructure complete relative to the i-structures, we just have to check that

$$(\mathfrak{B}_i, S_i) \equiv (\mathfrak{A}_i, S_i) \quad \text{for all } i \in I.$$

Fix $i \in I$ and let ψ be an $\mathcal{L}_i(S_i)$-sentence such that

$$(\mathfrak{A}_i, S_i) \models \psi.$$

Then $\psi_i \in \mathbf{I}(\mathfrak{A})$ and therefore

$$(\mathfrak{B}, S) \models \psi_i$$

which by Lemma A.3 is equivalent to

$$(\mathfrak{B}_i, S_i) \models \psi.$$

For the converse implication, assume that

$$(\mathfrak{B}_i, S_i) \vDash \psi$$

and suppose that $(\mathfrak{A}_i, S_i) \nvDash \psi$. Then $(\neg\psi)_i \in \mathbf{I}(\mathfrak{A})$, so

$$(\mathfrak{B}, S) \vDash (\neg\psi)_i$$

which by Lemma A.3 is equivalent to

$$(\mathfrak{B}_i, S_i) \vDash \neg\psi,$$

a contradiction. Hence, $(\mathfrak{A}_i, S_i) \vDash \psi$. We have shown that any model of $\mathbf{T}'$ is elementarily equivalent to $(\mathfrak{A}, S)$. Since $(\mathfrak{A}, S) \vDash \mathbf{T}'$ and $(\mathfrak{A}, S) \vDash \varphi(c_1, \ldots, c_m)$ we may conclude that $\mathbf{T}' \vDash \varphi(c_1, \ldots, c_m)$.

By the Finiteness Theorem and the definition of $\mathbf{T}'$, we obtain $\theta_1, \ldots, \theta_k \in \mathbf{D}(\mathfrak{S})$ and $(\lambda^1)_{i_1}, \ldots, (\lambda^n)_{i_n} \in \mathbf{I}(\mathfrak{A})$ such that

$$\mathbf{T} \vDash \bigwedge_{l=1}^{k} \theta_l \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j} \to \varphi(c_1 \ldots, c_m).$$

Since $\bigwedge_{l=1}^{k} \theta_l$ is quantifier-free, we have that

$$\neg \left(\bigwedge_{l=1}^{k} \theta_l \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j} \right) \in \mathbf{A}(\varphi)$$

and therefore

$$(\mathfrak{A}, S) \vDash \neg \left(\bigwedge_{l=1}^{k} \theta_l \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j} \right).$$

This is a contradiction since $(\mathfrak{A}, S)$ is a model of $\mathbf{T}'$, in particular

$$(\mathfrak{A}, S) \vDash \bigwedge_{l=1}^{k} \theta_l \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j} \ .$$

Therefore, the model $(\mathfrak{A}, S)$ of $\mathbf{S}$ cannot exist and $\mathbf{S}$ is inconsistent as claimed. This proves one implication of the theorem. Indeed, from $\mathbf{S} \vDash \bot$, by the Finiteness Theorem we find $\neg\psi_1, \ldots, \neg\psi_m \in \mathbf{A}(\varphi)$ such that

$$\mathbf{T} \vDash \varphi \to \bigvee_{q=1}^{m} \psi_q \ .$$

Hence, by definition of $\mathbf{A}(\varphi)$

$$\mathbf{T} \models \varphi \leftrightarrow \bigvee_{q=1}^{m} \psi_q \ .$$

We will now prove that relative quantifier elimination implies relative substructure completeness. We have to show that condition (5.1) implies $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$ for all $\mathbf{T}$-models $\mathfrak{A}$, $\mathfrak{B}$ with a common substructure $\mathfrak{S}$. Let $\varphi = \varphi(c_1, \ldots, c_m)$, with $c_1, \ldots, c_m \in S$, be an $\mathcal{L}(S)$-sentence which holds in $(\mathfrak{A}, S)$. By assumption we have that the $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_m)$ is $\mathbf{T}$-equivalent to a disjunction of $\mathcal{L}$-formulae of the form

$$\theta \wedge \bigwedge_{j=1}^{n} (\lambda^j)_{i_j}$$

where $n \in \mathbb{N}$, θ is quantifier-free and for $1 \leq j \leq n$, λ^j is an $\mathcal{L}_{i_j}$-formula for some $i_1, \ldots, i_n \in I$. Thus, at least one $\mathcal{L}(S)$-sentence of this form (where $x_1, \ldots, x_m$ are replaced by $c_1, \ldots, c_m$, respectively) holds in $(\mathfrak{A}, S)$. This is equivalent to $(\mathfrak{A}, S) \models \theta$ and $(\mathfrak{A}, S) \models (\lambda^j)_{i_j}$ for all $1 \leq j \leq n$. Hence, equivalent to $(\mathfrak{A}, S) \models \theta$ and $(\mathfrak{A}_{i_j}, S_{i_j}) \models \lambda^j$ for all $1 \leq j \leq n$ by (TR). From condition (5.1) we infer that the latter is equivalent to $(\mathfrak{B}_{i_j}, S_{i_j}) \models \lambda^j$ for all $1 \leq j \leq n$ which in turn is equivalent to $(\mathfrak{B}, S) \models (\lambda^j)_{i_j}$ for all $1 \leq j \leq n$ again by (TR). On the other hand, since θ is quantifier-free, $(\mathfrak{A}, S) \models \theta$ is equivalent to $(\mathfrak{B}, S) \models \theta$ (cf. Corollary 1.29). This shows that $(\mathfrak{A}, S) \models \varphi$ is equivalent to $(\mathfrak{B}, S) \models \varphi$ as we wanted. □

Appendix B

A universal axiom for associativity

In Remark 3.30 we claimed that there is a universal axiom for associativity which works in the case of the γ-valued hyperfields. We will provide a way to write such a sentence in this appendix.

Lemma B.1. *Let H be a canonical hypergroup. For all $x, y, z \in H$ we have that*

$$a \in (x+y)+z \iff (x+y) \cap (a-z) \neq \emptyset.$$

and

$$a \in x+(y+z) \iff (y+z) \cap (a-x) \neq \emptyset$$

Proof. If $a \in (x+y)+z$, then there exists $b \in x+y$ such that $a \in b+z$. By reversibility we obtain that $b \in a-z$, therefore $b \in (x+y) \cap (a-z) \neq \emptyset$.

If $b \in (x+y) \cap (a-z)$, then by reversibility $a \in b+z$, so

$$a \in \bigcup_{b' \in x+y} b'+z = (x+y)+z.$$

The second assertion is proved in a similar way. □

Corollary B.2. *Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. For all $x, y, z \in K$ we have that*

$$[a]_\gamma \in ([x]_\gamma + [y]_\gamma) + [z]_\gamma \iff [x]_\gamma + [y]_\gamma \subseteq [a]_\gamma - [z]_\gamma \text{ or } [a]_\gamma - [z]_\gamma \subseteq [x]_\gamma + [y]_\gamma$$

and

$$[a]_\gamma \in [x]_\gamma + ([y]_\gamma + [z]_\gamma) \iff [y]_\gamma + [z]_\gamma \subseteq [a]_\gamma - [x]_\gamma \text{ or } [a]_\gamma - [x]_\gamma \subseteq [y]_\gamma + [z]_\gamma \ .$$

Proof. By Proposition 3.26, $[x]_\gamma + [y]_\gamma$ and $[a]_\gamma - [z]_\gamma$ are ultrametric balls, therefore by Lemma 3.24 if they have non-empty intersection, then they are one contained in the other. Trivially, the converse is also true. By virtue of the previous lemma this completes the proof of the first equivalence. The second is similar. □

It is now not difficult to give a universal axiom for associativity which works in the case of the γ-valued hyperfields. Consider the axiom for associativity that we have given in Remark 3.30:

$$\forall x \forall y \forall z \forall t((\exists a(r_+(x,y,a) \wedge r_+(a,z,t))) \leftrightarrow (\exists b(r_+(y,z,b) \wedge r_+(x,b,t)))). \quad \text{(B.1)}$$

Now, the existential part

$$\exists a(r_+(x,y,a) \wedge r_+(a,z,t))$$

simply means that $t \in (x+y)+z$ and by Corollary B.2 we can replace it by the following universal formula

$$\forall a \forall a'((r_+(x,y,a) \to r_+(t,-z,a)) \vee r_+(t,-z,a') \to r_+(x,y,a'))$$

expressing the inclusions of the ultrametric balls $x+y$ and $t-z$.

Similarly, we can replace the other existential part in (B.1), which just means $t \in x+(y+z)$, by a universal formula. Using these replacements when needed, we obtain a universal $\mathcal{L}_{hf}$-sentence which is equivalent to the associativity of $+$ in $\mathcal{H}_\gamma(K)$ as contended.

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Uniwersytet Szczeciński

Instytut Matematyki

Imię i nazwisko / stopień: mgr Alessandro Linzi

Tytuł rozprawy doktorskiej **(czcionka pogrubiona): Algebraic hyperstructures in the model theory of valued fields**

promotor: stopień/tytuł naukowy/**imię i nazwisko** **prof. dr hab. Franz-Viktor Kuhlmann**
promotor pomocniczy: stopień/tytuł naukowy/ **imię i nazwisko -**
(jeżeli jest zatwierdzony uchwałą RW)

Streszczenie rozprawy doktorskiej w języku angielskim

In the literature, quantifier elimination for valued fields has been widely studied. It is well-known that algebraically closed valued fields admit quantifier elimination in their natural language. For *p*-adically closed fields MacIntyre showed that to eliminate quantifiers one has to extend the language of valued fields with the so-called power predicates. If we consider henselian valued fields of residue characteristic 0, then we have a result of Pas which states that quantifier elimination is achieved relative to the value group and the residue field; but for this an angular component map is needed. In 1991, Basarab introduced some structures which are associated to valued fields and relative to which it is possible to eliminate quantifiers for henselian valued fields of characteristic 0. Basarab's work was followed up by F.-V. Kuhlmann who introduced amc-structures. Later, Flenner noticed that Kuhlmann's approach can be simplified using what he called RV-structures. The main aim of this work is to describe a general framework where all these structures can be linked and better understood. In 1957 Krasner defined valued hyperfields. These are the structures that provide this general framework. Indeed, RV-structures are valued hyperfields. We extensively study the theory of valued hyperfields under a more general definition of valuation on a hyperfield than the one used by Krasner. Then we link the valued hyperfields associated to a valued field to several other structures which can be associated to a valued field. This investigation took into account the amc-structures as well as graded rings. We also study what happens to Krasner's valued hyperfields when an angular component map is present. We then link all the results on quantifier elimination for henselian valued fields of characteristic 0 using the technique of substructure completeness.

Data, podpis

30.03.2022 Alessandro Linzi

słowa kluczowe w języku polskim (odpowiedniki słów kluczowych w języku angielskim).

Ciało waluacji (valued field), hiperciało (hyperfield), eliminacja kwantyfikatorów (quantifier elimination), teoria modeli (model theory), hiperstruktury (hyperstructures)

Uniwersytet Szczeciński

Instytut Matematyki

Imię i nazwisko / stopień: mgr Alessandro Linzi

Tytuł rozprawy doktorskiej **(czcionka pogrubiona): Algebraic hyperstructures in the model theory of valued fields**

Tłumaczenie tytułu rozprawy na polski: **Algebraiczne hiperstruktury w teorii modeli ciał waluacji**

promotor: stopień/tytuł naukowy/**imię i nazwisko** **prof. dr hab. Franz-Viktor Kuhlmann**
promotor pomocniczy: stopień/tytuł naukowy/ **imię i nazwisko -**
(jeżeli jest zatwierdzony uchwałą RW)

Streszczenie rozprawy doktorskiej w języku polskim

Eliminacja kwantyfikatorów dla ciał z waluacją była szeroko badana w literaturze. Powszechnie znanym faktem jest, że algebraicznie domknięte ciała z waluacją mają własność eliminacji kwantyfikatorów w swoim naturalnym języku. Dla ciał p-adycznie domkniętych dla eliminacji kwantyfikatorów należy rozszerzyć język ciał z waluacją o tak zwane predykaty potęgowe, co zostało pokazane przez Macintyre'a. Rozważając ciała henselowskie z waluacją i ciałem reszt o charakterystyce 0, mamy wynik Pas'a, który mówi, iż w tym wypadku możemy otrzymać eliminację kwantyfikatorów po rozszerzeniu struktury o grupę wartości oraz ciało reszt, ale potrzebujemy do tego funkcji łukowej składowej. W 1991 Basarab zdefiniował struktury, o które możemy wzbogacić strukturę ciał z waluacją, celem uzyskania eliminacji kwantyfikatorów dla przypadku ciał henselowskich charakterystyki 0. Następnie, F.-V. Kuhlmann, podążając za wynikami Basarab'a, skonstruował amc-struktury. Natomiast Flenner zauważył, iż struktury wprowadzone przez F.-V. Kuhlmanna mogą zostać uproszczone, co zrobił wprowadzając RV-struktury. Głównym celem tej pracy jest opisanie ogólnej teorii, w której zostaną zbadane powyższe struktury i związki miedzy nimi. W 1957 Krasner zdefiniował hiperciała z waluacją, i to na nich bazuje ta ogólna teoria. Istotnie, struktury RV są hiperciałami z waluacją. W mojej pracy badam teorię hiperciał, używając bardziej ogólnej definicji waluacji na hiperciele, niż tej używanej przez Krasnera. Następnie przedstawiam związek hiperciał z waluacją stowarzyszonych z ciałami z waluacją z innymi strukturami, które mogą zostać stowarzyszone z ciałami z waluacją. W moich badaniach rozważałem amc-struktury oraz pierścienie z gradacją. Sprawdzam również co się dzieje z hiperciałami Krasnera z waluacją, gdy istnieje funkcja łukowej składowej. W końcu, łączę wszystkie wyniki na temat eliminacji kwantyfikatorów dla ciał henselowskich z waluacją o charakterystyce 0 używając techniki zupełności podstruktur.

Data, podpis

30.03.2022

słowa kluczowe w języku polskim (odpowiedniki słów kluczowych w języku angielskim).

Ciało waluacji (valued field), hiperciało (hyperfield), eliminacja kwantyfikatorów (quantifier elimination), teoria modeli (model theory), hiperstruktury (hyperstructures)