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Continuity of Roots and Values for Valued Fields

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Introduction

In this work, we will consider valued fields, that is, fields which are equipped with a mapping called *a valuation*. Any valuation v on a field K induces a topology on K. This gives rise to a notion of "closeness". Intuitively, we will say that two elements $a, b \in K$ are *close to each other* if the value $v(a-b)$ is *large*. This notion is then extended to polynomials in $K[x]$: we say that two polynomials are *close to each other* if their coefficients next to the respective powers of x are close to each other. This notion of closeness comes from the topology induced by the *Gauß extension* of the valuation from K to $K[x]$. With respect to that extension, the value of a polynomial is defined to be the minimum of the values of the coefficients.

The basic principle of root continuity states that the closeness of polynomials implies the closeness of their roots under a suitable pairing. This result has been proved e.g. in sources such as [5] and [15]. Further studies have shown that we can say much more about the connections between polynomials that are sufficiently close to each other. Moreover, the results on continuity of roots find applications in numerous areas of mathematics. One of the simplest applications is the result which states that the completion of a Henselian field is again Henselian (cf. Theorem 6.0.1), regardless of the rank of the valuation. In [2], Brink employs theorems on continuity of roots and continuity of factors of polynomials to present a general version of Hensel's Lemma. Theorem 32.20 from [15] employs a basic version of a root continuity result to study the behavior of irreducible factors of polynomials which are close to each other.

This dissertation aims to give a comprehensive overview of the results from the literature, as well as to improve and build on those results. An example of such an enhancement is Theorem 2.1.3 which is an improved version of the basic results from [5] and [15]. The basic results say that given any value ε, if two monic polynomials f and g are sufficiently close to each other, then their roots can be paired in such a way that the value of their difference is larger than ε. The improved result gives an explicit bound for the value $v(f-g)$ for which the aforementioned pairing of roots can be achieved.

This result also states that, possibly under additional assumptions, several invariants for f are the same as for g, including the degree, the minimal value of the roots and the value

$$\mathrm{kras}(f) = \max\{v(\alpha - \alpha') \mid \alpha \neq \alpha' \text{ are roots of } f\}.$$

When f has only one root, we take $\mathrm{kras}(f)$ to be the value of that root.

The above theorem, along with other basic results such as those from [1], [9] and [14], are presented in Chapter 2. In that chapter we also study other approaches to stating and proving root continuity. One of those approaches studies convergent nets of polynomials in Theorem 2.2.2 and Theorem 2.2.5. In both theorems we assume that a polynomial f is a limit of a net $(f_i)_{i \in I}$, where I is a directed set. If we now choose a root β_i of f_i for each $i \in I$, then the net $(\beta_i)_{i \in I}$ will contain a subnet convergent to some root of f. Conversely, every root of f is a limit of a suitable net $(\beta_i)_{i \in I}$ of roots of the converging polynomials.

Another approach to root continuity employs induction on the degree of the polynomial (Section 2.3, Theorem 2.3.2). The idea of the inductive method is as follows: consider two polynomials f and g such that $v(f - g)$ is sufficiently large, and find roots α of f and β of g which are close to each other. Then divide f by $x - \alpha$ and g by $x - \beta$ and work with the resulting polynomials in place of f and g. We then continue this method inductively, until we end up at linear polynomials. As a result, we obtain that there exists a pairing between the roots of f and g which are close to each other. However, the inductive method causes the bound for $v(f - g)$ to have factorial growth with respect to the degree of f. This makes it a relatively weak bound compared to the bounds in a number of other theorems given in this dissertation.

The method of proving root continuity that will be studied most thoroughly in this dissertation involves the notion of *Newton Polygons*. In Chapter 3 we will introduce this notion in full generality for points in the Cartesian product $\mathbb{R} \times \Gamma$. Here, Γ is an extension of an ordered Abelian group defined in Section 1.5, given by *the Hahn product* of sufficiently many copies of $\mathbb{R}$. Any ordered Abelian group can be embedded in a group of this form (see Theorem 1.5.1). This provides the necessary generality when we work with elements in the value group of a valued field which is in itself an ordered Abelian group. We note, however, that for our purposes it is enough to work with the divisible hull of the value group.

In Section 3.2, we will employ the previously introduced general notion of a Newton Polygon to define the Newton Polygon of a polynomial. We give a detailed proof of the known result which states that there is a one-to-one correspondence between the slopes of the Newton Polygon and the values of roots of the polynomial (Theorem 3.2.2).

The results from Chapter 3 will be employed in Chapter 4 to study connections between the Newton Polygons of polynomials f and g which are close to each other. This is done in particular in Theorem 4.1.1 which states that the respective Newton Polygons of f and g will coincide along a certain real interval. The left end of this interval depends on the value $v(f-g)$ and, if 0 is a root of f, on the multiplicity of 0. The right end of this interval will always be located at the degree of f. Note that in Theorem 4.1.1 we will not assume that g is monic nor that it is of the same degree as f. To our knowledge, a result in this generality has not been stated in the literature yet.

Theorem 4.1.1 is then applied in Theorem 4.1.5 which tells us about how many of the respective roots of f and g have value equal to a given value γ, and how many will have value greater than γ. This theorem gives more detailed information than the results in [6]. Moreover, it can readily be adapted to the case where f and g are polynomials over two different valued fields whose respective value groups are contained in a common ordered Abelian group. The aforementioned Theorem 4.1.5 is a result that can be seen as *continuity of values of roots*. At the end of Section 4.1, we have a close look at another statement that can also be understood as continuity of values, and we disprove this statement (see Example 4.1.9).

The results from Section 4.1 are used in Section 4.2 to formulate root continuity theorems. We state results known from literature, as well as their alterations which can be applied in cases that require omitting some of the assumptions of the original statements. As an example, Theorem 4.2.2 is a result similar to one that can be found in [2]. Theorem 4.2.2 states that if the value $v(f-g)$ is large enough, then there exists a pairing between the roots of f and g which are close to each other. The bound given in Theorem 4.2.2 is not as precise as the one given in the original result of [2]. However, unlike that result, the theorem does not assume that the polynomials in question are monic or of the same degree, nor that their coefficients are integral. This theorem is then employed to prove Theorem 4.2.4 which is a generalization of a result from [1] (see Theorem 4.2.5). Under the same assumptions as Theorem 4.2.2, it states that for each root α of f, the open ultrametric ball with large enough radius and centered at α contains the same number of roots of f and of g (counted with multiplicity). We also state results from [6] and [7] and compare them with Theorem 4.2.2. While possibly using a stronger bound for $v(f-g)$, our theorem does not assume that the polynomials in question are of the same degree. Moreover, the bound for the ultrametric distance between a root α of f and a root β of g is stronger than the bound given in [6] and [7] as soon as α is not a simple root of f. Finally, in Section

4.3 we study and aim to generalize results from [2].

In Chapter 5 we employ the root continuity theorems proved for polynomials to formulate analogous results for roots and poles of rational functions. To this end, we define an ultrametric on the rational function field which does not come from a valuation. We then compare this ultrametric with the one coming from the canonical extension of the Gauß valuation from the polynomial ring to the rational function field.

Finally, in Chapter 6 we present applications of the root continuity theorems from the previous chapters. There, we study relations between other attributes of polynomials which are close to each other. For example, we state connections between the irreducible factors of the polynomials and the extensions generated by their roots. Such a statement can be found in Theorem 6.1.3 which originates from [15], and in Theorem 6.2.5 which is a version of Theorem 6.1.3 for polynomials that need not be separable. Both results state that under a suitable pairing, the extensions generated by the roots of the irreducible factors of the polynomials in question are isomorphic (either over the ground field, or over its henselization), and that the splitting fields of each pair of factors are the same. Moreover, we prove that polynomials which are close to each other define extensions with the same ramification theoretical invariants (Theorem 6.3.5).

A significant number of results from this dissertation has been included in [3], a paper written by the author together with co-authors P. Szewczyk and F.-V. Kuhlmann.

Chapter 1

Notation and background

In this dissertation, we will denote:

- by $\mathbb{R}$ the field of real numbers,
- by $\mathbb{Q}$ the field of rational numbers,
- by $\mathbb{Z}$ the ring of integers,
- by $\mathbb{N}$ the set of positive integers,
- by $\mathbb{N}_0$ the set of non-negative integers,
- by $\mathbb{P}$ the set of prime numbers,
- by $\mathbb{F}_p$ the finite field with p elements for $p \in \mathbb{P}$.

1.1 Valuation-theoretical background

1.1.1 Basics of valuation theory

In this section, we will introduce the notation and facts on valuation theory which will be used in the later parts of the dissertation. For further background on valuation theory we refer the reader to sources such as [4], [5], [9], and [11, Chapter 2].

Let $(K, +, \cdot, 0, 1)$ be a field. Consider a mapping v from K to $\Gamma \cup \{\infty\}$ for some ordered Abelian group $(\Gamma, +, \leq)$. Here, ∞ is the symbol for an element greater than every element of Γ that satisfies

$$\infty = \infty + \infty = \gamma + \infty = \infty + \gamma \quad \text{for all } \gamma \in \Gamma.$$

We say that v is a *valuation on K* if for all $a, b \in K$ we have that

($v1$) $v(a) = \infty \iff a = 0$,

($v2$) $v(a+b) \geq \min\{v(a), v(b)\}$ (*ultrametric triangle law*),

($v3$) $v(a \cdot b) = v(a) + v(b)$.

We will then call (K, v) *a valued field* and the element $v(a) \in \Gamma$ *the value of* a. If no confusion arises, we will write va in place of $v(a)$.

Denote $K \setminus \{0\}$ by $K^\times$. From axiom ($v3$) we obtain that for all $a \in K$ and $n \in \mathbb{N}$ we have that $v(a^n) = nva$, where nva is the n-fold sum of the element $va \in \Gamma$. Moreover, $v\frac{a}{b} = va - vb$ for all $a \in K$, $b \in K^\times$. In particular, $v1 = v\frac{1}{1} = v1 - v1 = 0$ and so $v(a^{-1}) = -va$ for all $a \in K^\times$. If $\zeta \in K$ is such that $\zeta^n = 1$ for some $n \in \mathbb{N}$, then $v(\zeta^n) = nv\zeta = 0$. Since Γ is ordered and thus contains no nontrivial torsion elements, we have that $v\zeta = 0$. In particular, $v(-1) = 0$ and so $v(-a) = v(-1) + va = va$ for all $a \in K$.

Applying axiom ($v2$) inductively yields

$$v\left(\sum_{i=1}^{n} a_i\right) \geq \min_{1 \leq i \leq n} va_i$$

for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in K$.

This in particular implies that every natural number n, taken as the element of K defined as the n-fold sum of 1, has a non-negative value under any valuation on K. Indeed, $vn = v(1 + \ldots + 1) \geq v1 = 0$. Since $vn = v(-n)$, also every integer in K has non-negative value.

We claim that if $va < vb$, then $v(a+b) = va$. Indeed, if we had $v(a+b) > va$, then

$$va = v((a+b) - b) \geq \min\{v(a+b), vb\} > va,$$

which gives us a contradiction.

Let (K, v) be an arbitrary valued field. We define

$$vK := v(K^\times).$$

Note that vK with the restictions of the operation $+$ and the ordering $\leq$ is an ordered subgroup of the group Γ. We will call it *the value group of* (K, v).

An integral domain R is called *a valuation ring* if for every a in the field of fractions $\mathrm{Quot}(R)$ either $a \in R$ or $a^{-1} \in R$. Observe that the set

$$\mathcal{O}_K := \{a \in K \mid va \geq 0\}$$

is a valuation ring. It will be called *the valuation ring of* (K, v).

We define the set

$$\mathfrak{M}_K := \{a \in K \mid va > 0\}$$

to be *the valuation ideal of* (K, v). Note that $\mathfrak{M}_K$ is indeed an ideal of $\mathcal{O}_K$. Since $\mathcal{O}_K \setminus \mathfrak{M}_K$ is precisely the set of all units in $\mathcal{O}_K$, we also have that $\mathfrak{M}_K$ is the unique maximal ideal of $\mathcal{O}_K$. We may thus define the field

$$Kv := \mathcal{O}_K / \mathfrak{M}_K$$

and call it *the reside field of* (K, v). The canonical epimorphism $\mathcal{O}_K \to Kv$ is called *the residue map of* (K, v). For $a \in \mathcal{O}_K$, the image $a + \mathfrak{M}_K$ will be denoted by av and called *the residue of* a. When considering polynomials over K, it will be useful to employ the following notation. For a polynomial

$$f(x) := \sum_{i=0}^{n} a_i x^i \in \mathcal{O}_K[x],$$

we will write

$$(fv)(x) := \sum_{i=0}^{n} (a_i v) x^i \in (Kv)[x] \tag{1.1}$$

and call it *the reduction of* f.

Let v_1 and v_2 be two valuations on K. We say that v_1 and v_2 are *equivalent* if there is an order-preserving isomorphism $\varphi : v_1 K \to v_2 K$ such that $v_2 a = \varphi(v_1 a)$ for all $a \in K^{\times}$. Observe that valuation rings of equivalent valuations are equal. If no confusion arises, from now on we will identify equivalent valuations.

1.1.2 Examples

In this section, we introduce a number of examples of valued fields which we will employ further in the dissertation. The value groups and residue fields of the fields below are considered up to isomorphism.

Example 1.1.1. The simplest example of a valuation on K is the trivial valuation. Here, we set $v0 = \infty$ and $va = 0$ for all $a \in K^{\times}$. In this case, $vK = \{0\}$ and $Kv = K$. Note that finite fields admit only the trivial valuation, since every nonzero element a of such a field satisfies $a^n = 1$ for some $n \in \mathbb{N}$.

Example 1.1.2. Another example that will be commonly used is the p-adic valuation on $\mathbb{Q}$. For $p \in \mathbb{P}$, we write every element $\frac{a}{b} \in \mathbb{Q} \setminus \{0\}$ as $\frac{a}{b} = p^k \cdot \frac{a'}{b'}$, where $k \in \mathbb{Z}$ and $a', b' \in \mathbb{Z}$ are not divisible by p. We then set $v\left(\frac{a}{b}\right) := k$. Here, we have that $v\mathbb{Q} = \mathbb{Z}$ and $\mathbb{Q}v = \mathbb{F}_p$.

If we have a mapping v on an integral domain R which satisfies axioms ($v1$)–($v3$), we will also call v a *valuation on* R and the pair (R, v) a valued ring. The valuation v can then be extended canonically to a valuation on the field of fractions $K := \text{Quot}(R)$ by setting

$$v\left(\frac{a}{b}\right) := va - vb.$$

In the previous example, it was possible to consider first the valuation v on $\mathbb{Z}$ and then extend it canonically to $\mathbb{Q}$ to obtain the p-adic valuation on $\mathbb{Q}$.

Example 1.1.3. Take an element t transcendental over K and consider the ring

$$K[t] := \left\{ \sum_{i=0}^{n} a_i t^i \mid n \in \mathbb{N}_0,\ a_i \in K \right\}.$$

We define

$$0 \neq f(t) = \sum_{i=0}^{n} a_i t^i \mapsto vf := \min\{i \in \{0, \ldots, n\} \mid a_i \neq 0\}.$$

We then extend the mapping v to the field $K(t) = \text{Quot}(K[t])$ canonically. This is called the *t-adic valuation*. We may further extend v to the *field of formal Laurent series*

$$K((t)) := \left\{ \sum_{i=N}^{\infty} a_i t^i \mid N \in \mathbb{Z},\ a_i \in K \right\}$$

by setting

$$v\left(\sum_{i=N}^{\infty} a_i t^i\right) := \min\{i \in \mathbb{Z} \mid a_i \neq 0\}.$$

In this case, $vK(t) = vK((t)) = \mathbb{Z}$ and $K(t)v = K((t))v = K$.

Example 1.1.4. The most important example of a valuation for us will be the *Gauß valuation* on the polynomial ring $K[x]$. Take a valued field (K, v) and let x be an independent variable. We define a valuation on $K[x]$, which we will once again denote by v, in the following manner:

$$v\left(\sum_{i=0}^{n} a_i x^i\right) := \min_{0 \leq i \leq n} va_i.$$

This valuation is also extended canonically to the field $K(x) = \text{Quot}(K[x])$. We then have that $vK(x) = vK$ and $K(x)v = Kv(xv)$.

1.2 Ultrametric spaces

Let X be a nonempty set and $(T, \leq)$ a totally ordered set. We define ∞ to be an element strictly greater than every element in T. We say that a mapping $u : X \times X \to T$ is *an ultrametric* if the following conditions hold for all $a, b, c \in X$ and $t \in T$:

$(u1)$ $u(a, b) = \infty \iff a = b$,

$(u2)$ $u(a, b) = u(b, a)$,

$(u3)$ $u(a, c) \geq \min\{u(a, b), u(b, c)\}$ *(ultrametric triangle law).*

In this case, we say that the pair (X, u) is *an ultrametric space.*

Example 1.2.1. Let (K, v) be a valued field and set $u(a, b) := v(a - b)$ for $a, b \in K$. Then (K, u) is an ultrametric space. However, not every ultrametric comes from a valuation (see Example 5.1.3 and Corollary 5.1.4).

As was the case with valuations, in an ultrametric triangle at least two sides are equal. In other words, if $u(a, b) < u(b, c)$, then

$$u(a, c) = \min\{u(a, b), u(b, c)\}.$$

We can prove this property in the same manner as the analogous property for valuations.

Similarly to the case of valuations, we can also apply axiom $(u3)$ inductively to obtain that the ultrametric triangle law holds for more than three points.

Definition 1.2.2. Let (X, u) be an ultrametric space, and let T be the ordered set connected with this space. Take $a \in X$, $t \in T$. We define the set

$$B_t(a) := \{b \in X \mid u(a, b) \geq t\}$$

and call it *the (closed) ultrametric ball of radius t centered at a.*

Similarly, we define the set

$$B^\circ_t(a) := \{b \in X \mid u(a, b) > t\}$$

and call it *the open ultrametric ball of radius t centered at a.*

Take $a, b, c \in X$, and $t \in T$. Observe that for all $a, c \in B^\circ_t(b)$ we have that $u(a, c) > t$ since

$$u(a, c) \geq \min\{u(a, b), u(b, c)\} > t. \tag{1.2}$$

Moreover, we claim that if $u(a, b) > t$, then $B_t^\circ(a) = B_t^\circ(b)$. In other words, every element in an open ultrametric ball is the center of that ball. Indeed, condition $u(a, b) > t$ implies that $a \in B_t^\circ(b)$. If $c \in B_t^\circ(b)$, then by (1.2) we have that $u(a, c) > t$. This means that $c \in B_t^\circ(a)$, which implies that $B_t^\circ(b) \subseteq B_t^\circ(a)$. The reverse inclusion is proved analogously.

Furthermore, two open ultrametric balls are either disjoint or comparable by inclusion. Indeed, take $t_1, t_2 \in T$ such that $t_1 \leq t_2$. If $c \in B_{t_1}^\circ(a) \cap B_{t_2}^\circ(b)$, then

$$u(a, b) \geq \min\{u(a, c), u(c, b)\} > t_1,$$

and so by what we have proved in the paragraph above, we obtain that

$$B_{t_1}^\circ(a) = B_{t_1}^\circ(b) \subseteq B_{t_2}^\circ(b).$$

The above arguments can be applied with "$\geq$" in place of "$>$" to obtain analogous results for closed ultrametric balls.

1.3 Fields and field extensions

Take a field K and choose an algebraic closure $\widetilde{K}$ of K.

A polynomial $f \in K[x]$ will be called *separable* if it has only *simple roots*, that is, roots of multiplicity 1. An element $\alpha \in \widetilde{K}$ will be called *separable over* K if it is a root of a separable polynomial over K. Similarly, an algebraic extension $L|K$ will be called *separable* if each element in L is separable over K. The set consisting of all elements in $\widetilde{K}$ separable over K is a field, called the *separable-algebraic closure*, which we will denote by K^{sep}. If an algebraic extension $L|K$ (or polynomial f or element α) is not separable, then we will call it *inseparable*. If f only admits one root, then it will be called *purely inseparable*. Similarly, $L|K$ is *purely inseparable* if each element $a \in L$ is a root of a purely inseparable polynomial over K. Observe that in our notation, linear polynomials are both separable and purely inseparable.

Let $L|K$ and $F|K$ be algebraic extensions of K. We say that elements $\alpha_1, \ldots, \alpha_n \in L$ are *K-linearly independent* if for every $c_1, \ldots, c_n \in K$, the condition

$$\sum_{i=1}^{n} c_i \alpha_i = 0$$

implies that $c_i = 0$ for $1 \leq i \leq n$. We say that $L|K$ is *linearly disjoint from $F|K$* if for every $n \in \mathbb{N}$ and every choice of K-linearly independent elements $\alpha_1, \ldots, \alpha_n \in L$, these elements will also be F-linearly independent. We will now show that this relation is symmetrical. Assume that $L|K$ is linearly disjoint from $F|K$ and suppose that there exist $\alpha_1, \dots, \alpha_n \in L$ and K-linearly independent elements $\beta_1, \dots, \beta_n \in F$ such that

$$\alpha_1\beta_1 + \ldots + \alpha_n\beta_n = 0. \tag{1.3}$$

This means that the elements α_i are not F-linearly independent, which by our assumption implies that they are also not K-linearly independent. Without loss of generality, we may assume that $\alpha_1, \dots, \alpha_m$ are K-linearly independent for some $m < n$ and that there exist $c_{ij} \in K$ such that

$$\alpha_i = \sum_{j=1}^{m} c_{ij}\alpha_j \quad \text{for } m < i \leq n.$$

We combine the above equation with (1.3) to obtain

$$\sum_{j=1}^{m} \alpha_j\beta_j + \sum_{i=m+1}^{n} \left(\sum_{j=1}^{m} c_{ij}\alpha_j \right) \beta_i = 0.$$

Reorganizing the terms, we obtain that

$$\sum_{j=1}^{m} \left(\beta_j + \sum_{i=m+1}^{n} c_{ij}\beta_i \right) \alpha_j = 0.$$

The elements β_i were assumed to be K-linearly independent, therefore the coefficient next to each element α_j is nonzero. This means that the elements $\alpha_1, \dots, \alpha_m$ are not F-linearly independent. On the other hand, they were assumed to be K-linearly independent, which contradicts our assumption that $L|K$ is linearly disjoint from $F|K$.

We now see that $L|K$ is linearly disjoint from $F|K$ if and only if $F|K$ is linearly disjoint from $L|K$. In view of this symmetry, we will say that in this case L and F are *linearly disjoint over* K.

Let L and F be arbitrary algebraic extensions of K. We define *the compositum of* L *and* F to be the smallest subfield of $\widetilde{K}$ that contains both L and F and we denote it by $L.F$.

Assume that the extension $L|K$ is finite and consider a K-basis $\mathcal{B}$ of L, that is, a maximal set of K-linearly independent elements in L. If L and F are K-linearly disjoint, then the elements of $\mathcal{B}$ remain F-linearly independent. Hence, $\mathcal{B}$ is also an F-basis of $L.F$. This means that $[L : K] = [L.F : F]$. Note that this equality holds also if $[L : K]$ is infinite.

Let L be a normal algebraic extension of K. That is, we assume that every irreducible polynomial over K which has a root in L, splits into linear factors in L. We will denote by $\mathrm{Gal}(L|K)$ the set of automorphisms of L over K, that is, automorphisms which leave K elementwise fixed. Note that we will be using this notation without assuming $L|K$ to be separable. In the particular case where $L = K^{\mathrm{sep}}$, we will write

$$\mathrm{Gal}\,K := \mathrm{Gal}(K^{\mathrm{sep}}|K).$$

If $\sigma \in \mathrm{Gal}\,L|K$ and $\alpha \in L$, then we will write $\sigma\alpha$ in place of $\sigma(\alpha)$.

Let $L|K$ be an arbitrary field extension and $\alpha \in L$ algebraic over K. Let $f \in K[x]$ be the minimal polynomial of α over K, that is, f is a monic and irreducible polynomial over K which admits α as a root. Then for all $\sigma \in \mathrm{Gal}(L|K)$, $\sigma\alpha$ is a root of f. Conversely, every root of f is of the form $\sigma\alpha$ for some $\sigma \in \mathrm{Gal}(L|K)$.

1.4 Valued field extensions

Let $L|K$ be a field extension, where (L, w) and (K, v) are valued fields. We say that w *extends* v (or w *is an extension of* v) *from* K *to* L if $w|_K = v$. In this case, we will also say that $(L, w)|(K, v)$ is a *valued field extension*. We will commonly use the symbol v for the valuation on both K and L, denoting by $(L|K, v)$ the respective valued field extension. The following theorem is a consequence of *Chevalley's Extension Theorem* ([5, Theorem 3.1.1]).

Theorem 1.4.1 (Theorem 3.1.2 from [5]). *Let (K, v) be a valued field and $L|K$ any field extension. Then there exists an extension of v from K to L.*

Consider a valued field extension $(L|K, v)$. We will identify vK and Kv with their natural embeddings in vL and Lv, respectively. In this sense, vK is a subgroup of vL and Kv is a subfield of Lv.

The *ramification index* of $(L|K, v)$ is $\mathrm{e}(L|K, v) = (vL : vK)$, and the *inertia degree* is $\mathrm{f}\,(L|K, v) := [Lv : Kv]$. If $\mathrm{e}(L|K, v) = 1$ and $\mathrm{f}\,(L|K, v) = 1$, then we say that the extension $(L|K, v)$ is *immediate*.

It is a well-known fact (see e.g. [5, Corollary 3.2.3]) that if $[L : K] < \infty$, then also $\mathrm{e}(L|K, v) < \infty$, $\mathrm{f}\,(L|K, v) < \infty$, and

$$\mathrm{e}(L|K, v) \cdot \mathrm{f}\,(L|K, v) \leq [L : K]. \tag{1.4}$$

In Section 6.3, we will look into a more general form of the above inequality.

For our purposes, we will fix an algebraic closure $\widetilde{K}$ of K and extend v from K to $\widetilde{K}$. We will denote this extended valuation by v as well. We then extend v once again to the valuation v on $\widetilde{K}(x)$ by means of the Gauß valuation from Example 1.1.4.

This extension will allow us to determine a correspondence between the roots of a polynomial f and the roots of its reduction fv. Note that $f \in \mathcal{O}_{K(x)}$ as an element of $K(x)$ if and only if $f \in \mathcal{O}_K[x]$ as a polynomial over K. In this case, the definition of fv in (1.1) coincides with the definition of the residue of f as an element of the valued field $K(x)$. Assume that f is a monic polynomial in $\mathcal{O}_K[x]$. Then all the roots of f have non-negative value (cf. Lemma 3.2.4). Since the residue map is a homomorphism from $\mathcal{O}_{\widetilde{K}(x)}$ to $\widetilde{K}(x)v = \widetilde{K}v(xv)$, we have that

$$f(x) = \prod_{i=1}^{n}(x - \alpha_i) \Rightarrow (fv)(xv) = \prod_{i=1}^{n}(xv - (\alpha_i v)).$$

This means that for every root α_i of f there is a corresponding root $\alpha_i v$ of fv. Conversely, for every root ζ of fv, there is a root α of f such that $\alpha v = \zeta$.

We claim that if $(K, v) \subseteq (L, v) \subseteq (\widetilde{K}, v)$, then vL/vK is a torsion group and the extension $Lv|Kv$ is algebraic. Indeed, take any element $a \in L$, then $[K(a) : K] < \infty$. By (1.4), $vK(a)/vK$ is a finite group, hence there exists $n \in \mathbb{N}$ such that $nva \in vK$. Therefore, vL/vK is a torsion group. Similarly, if $a \in \mathcal{O}_L$, then the extension $K(a)v|Kv$ is finite by (1.4). Hence, av is algebraic over Kv, and so $Lv|Kv$ is algebraic.

We say that vK is *divisible* if for every $a \in vK$ and $n \in \mathbb{N}$ there exists $b \in G$ such that $nb = a$. Since vK is torsion-free, it admits a divisible extension group. This group has the universal property that it admits a unique embedding in every other divisible extension group, so it is unique up to isomorphism. We will call it *the divisible hull of* vK and denote it by $\widetilde{vK}$.

We will now show that $v\widetilde{K}$ is the divisible hull of vK and that $\widetilde{K}v$ is the algebraic closure of Kv. By what we have shown above, $v\widetilde{K}/vK$ is torsion and $\widetilde{K}v|Kv$ is algebraic.

Take any element $\gamma \in \widetilde{vK}$, that is, $n\gamma \in vK$ for some $n \in \mathbb{N}$. Take $a \in K$ such that $va = n\gamma$. Then for any $b \in \widetilde{K}$ such that $b^n = a$ we have that $vb = \gamma$, hence $\gamma \in v\widetilde{K}$.

Similarly, take $\zeta \in \widetilde{Kv}$. Let $f \in \mathcal{O}_K[x]$ be any monic polynomial whose reduction fv is the minimal polynomial of ζ over Kv. Then there exists a root $b \in \mathcal{O}_{\widetilde{K}}$ such that $bv = \zeta$. Therefore, $\zeta \in \widetilde{K}v$.

This shows that $v\widetilde{K} = \widetilde{vK}$ and $\widetilde{K}v = \widetilde{Kv}$.

In the further parts of this dissertation, we will commonly use expressions such as "for every $\varepsilon \in vK$ large enough" and "there exists $\delta \in vK$ large enough". However, in some proofs and applications we may be specifying elements ε and δ which are in $v\widetilde{K}$ and not necessarily in vK. This poses no threat to the generality of our theorems, since vK is cofinal in $\widetilde{vK} = v\widetilde{K}$.

Hence, we may replace $\varepsilon \in v\widetilde{K}$ and $\delta \in v\widetilde{K}$ with some elements $vK \ni \varepsilon' \geq \varepsilon$ and $vK \ni \delta' \geq \delta$.

In a similar vein, we will commonly use quotients $\frac{\delta}{n}$ for $\delta \in vK$ and $n \in \mathbb{N}$, working in $v\widetilde{K}$, even if the value group vK which we are considering is not divisible.

A valued field (K, v) is called *Henselian* if the extension of v to $\widetilde{K}$ is unique, or equivalently, if it satisfies the assertion of *Hensel's Lemma* (see e.g. [4, Corollary 16.6], [5, Theorem 4.1.3]):

Lemma 1.4.2. *Take $f \in \mathcal{O}_K[x]$. If fv has a simple root $\zeta \in Kv$, then f admits a root $\alpha \in \mathcal{O}_K$ such that $\alpha v = \zeta$.*

Every valued field (K, v) admits a minimal algebraic extension in $(\widetilde{K}, v)$ that satisfies Hensel's Lemma. This extension is unique up to isomorphism and is called the *henselization* of (K, v). We denote it by K^h or $(K, v)^h$.

1.5 Abelian groups

1.5.1 Valued groups

Let $(G, 0, +)$ be an Abelian group and consider a nonempty ordered set $(T, \leq)$. Denote by ∞ an element strictly greater than every element in T. A map

$$v : G \ni a \mapsto va \in T \cup \{\infty\}$$

is called *a valuation* if for all $a, b \in G$,

$(v1)$ $va = \infty \iff a = 0$,

$(v2)$ $v(a - b) \geq \min\{va, vb\}$.

We call the pair (G, v) *a valued group*. We write $vG := \{va \mid a \in G \setminus \{0\}\}$. Observe that taking $a = 0$ in $(v2)$ and replacing b once by a and once by $-a$, we obtain $v(-a) \geq va \geq v(-a)$, hence $va = v(-a)$, as was the case for valued fields. This means that we can apply axiom $(v2)$ inductively to obtain that $v(na) \geq va$, where na denotes the n-fold sum of a for some $n \in \mathbb{N}$.

We will now assume that G is an ordered Abelian group, that is, we have a total order $\leq$ on G. For an element $a \in G$ we write $|a| := \max\{a, -a\}$.

Two elements $a, b \in G$ are called *Archimedean equivalent* if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. As the name suggests, this is an equivalence relation. For an element a, we denote by va the Archimedean equivalence class of a. The set of all such equivalence classes can be ordered in the following manner: $va = vb$ if a and b are Archimedean equivalent, and $va < vb$ if b is Archimedean smaller than a, that is, $|a| > |b|$ and a and b are not Archimedean equivalent. Moreover, we define $v0 := \infty$ to be the maximal element in the set of equivalence classes. Then the mapping $a \mapsto va$ is a valuation on G, called *the natural valuation*. The order type of vG is called *the principal rank of* G. The group G is called *Archimedean* if the natural valuation on G is trivial, that is, if the principal rank of G is 1.

1.5.2 Hahn products

Take a nonempty totally ordered set I and a family of Abelian groups G_i, $i \in I$. Consider the group $G := \prod_{i\in I} G_i$ with component-wise addition and take $a = (a_i)_{i\in I} \in G$. We define *the support of* a to be the set

$$\mathrm{supp}(a) := \{i \in I \mid a_i \neq 0\}.$$

Then *the Hahn product* is the set

$$H := \mathbf{H}_{i\in I} G_i := \{a \in G \mid \mathrm{supp}(a) \text{ is well-ordered}\}.$$

Observe that $\mathrm{supp}(a + b) \subseteq \mathrm{supp}(a) \cup \mathrm{supp}(b)$. Thus if $\mathrm{supp}(a)$ and $\mathrm{supp}(b)$ are well-ordered, then so is $\mathrm{supp}(a + b)$. This means that H is a subgroup of G. We can define a valuation v on H as follows:

$$H \setminus \{0\} \ni a \mapsto va := \min \mathrm{supp}(a) \in I.$$

If in addition the groups G_i are ordered, then we may define the *lexicographic order* on H in the following manner: take $a = (a_i)_{i\in I} \in H$, we then say that $a > 0$ if $a_{va} > 0$. In this sense, H becomes an ordered Abelian group. If all the groups G_i are Archimedean, then the valuation on H coincides with the natural valuation from Section 1.5.1. On the other hand, the following theorem (see e.g. [12, Chapter I, Sect. 5, Satz 3]) allows us to identify an arbitrary ordered Abelian group with a subgroup of a suitable Hahn product.

Theorem 1.5.1 (Hahn's Embedding Theorem)**.** *Let G be an ordered Abelian group and let v be the natural valuation on G. Then G can be embedded into $\mathbf{H}_{i\in vG}\mathbb{R}$, where $\mathbb{R}$ is seen as the additive group of the real numbers.*

1.6 The Taylor expansion

In this section we introduce the "characteristic blind" Taylor expansion for polynomials. This means that it does not contain any denominators of natural numbers which in positive characteristic could be equal to 0. Throughout this section, we assume (K, v) to be an arbitrary valued field and we take

$$f(x) = \sum_{i=0}^{n} a_i x^i \in K[x].$$

In this section, the valuation on $K[x]$ will be the Gauß valuation, once again denoted by v.

The Taylor expansion of f employs the notion of *Hasse–Schmidt derivatives*, that is, the following expressions:

$$\partial_i f(x) := \sum_{j=i}^{n} a_j \binom{j}{i} x^{j-i} = \sum_{j=0}^{n-i} a_{j+i} \binom{j+i}{i} x^j, \quad 0 \le i \le n. \tag{1.5}$$

The polynomials $\partial_i f$ yield the following polynomial identity, which is called the *characteristic blind Taylor expansion for the polynomial f*:

$$f(x+y) = \sum_{0 \le i \le n} \partial_i f(y) x^i \,. \tag{1.6}$$

Recall from Section 1.1.1 that the value of every integer is non-negative. By the definition of the Gauß valuation, for every $j \in \{0, \ldots, n\}$ we have that

$$v\partial_i f = \min_{i \le j \le n} va_j \binom{j}{i} \ge \min_{i \le j \le n} va_j \ge \min_{0 \le j \le n} va_j = vf.$$

Lemma 1.6.1. *Take $c \in K$ and a polynomial $f \in K[x]$ of degree n. Then*

$$v(\partial_i f(c)) \ge vf + \min\{0, (n-i)vc\} \quad \textit{for} \quad 0 \le i \le n.$$

Proof. We will employ Equation (1.5). If $vc \ge 0$, then we have that

$$\begin{aligned} v\partial_i f(c) &\ge \min_{i \le j \le n} \left\{ va_j + v\binom{j}{i} + (j-i)vc \right\} \ge \min_{i \le j \le n} va_j = v\partial_i f \\ &\ge vf = vf + \min\{0, (n-i)vc\}. \end{aligned}$$

If $vc < 0$, then we have that

$$\begin{aligned} v\partial_i f(c) &\ge \min_{i \le j \le n} \left\{ va_j + v\binom{j}{i} + (j-i)vc \right\} \ge \min_{i \le j \le n} va_j + (n-i)vc \\ &= v\partial_i f + (n-i)vc \ge vf + (n-i)vc = vf + \min\{0, (n-i)vc\}. \end{aligned}$$

□

Definition 1.6.2. For $f \in K[x]$ and $c \in K$, we set $f_c(x) := f(x+c)$.

Lemma 1.6.3. *Take $c \in K$. Given polynomials $f, g \in K[x]$, we have that:*

$$v(f_c - g_c) \geq v(f-g) + \min\{0, \deg(f-g)vc\}.$$

In particular, if f and g are monic polynomials of degree n, then

$$v(f_c - g_c) \geq v(f-g) + \min\{0, (n-1)vc\}.$$

Proof. Set $h(x) := f(x) - g(x)$ and $r := \deg h$. Then from (1.6) we obtain:

$$v(f_c - g_c) = vh(x+c) = v\left(\sum_{0 \leq i < r} \partial_i h(c) x^i\right) = \min_{0 \leq i < r} v(\partial_i h(c)).$$

Now we use Lemma 1.6.1 to conclude:

$$\min_{0 \leq i < r} v(\partial_i h(c)) \geq \min_{0 \leq i < r} (vh + \min\{0, (r-i)vc\}) = vh + \min\{0, rvc\}.$$

If $\deg g = \deg f = n$ and both f and g are monic, then $r \leq n-1$ and so the above value is greater than or equal to $vh + \min\{0, (n-1)vc\}$. □

Chapter 2

Overview and improvements

In this chapter we present a number of results on continuity of roots. To the results from the literature that we cite in the dissertation we will also present improvements, either by generalizing the original statement, or by providing further results. Moreover, in Section 2.3 we present an additional method of proving root continuity theorems.

In this chapter, (K, v) will be an arbitrary valued field. **Unless specified otherwise, the valuation on $K[x]$ in this chapter and in all further chapters will be the Gauß valuation, which we will once again denote by v.** This means that polynomials $f, g \in K[x]$ are "close to each other" if their coefficients are "close to each other".

Throughout this dissertation, we will use the following notation for polynomials $f, g \in K[x]$:

$$\left.\begin{array}{ll} f(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{i=1}^{n}(x - \alpha_i), & \alpha_i \in \widetilde{K},\ a_i \in K, \\ g(x) = \sum_{i=0}^{m} b_i x^i = b_m \prod_{i=1}^{m}(x - \beta_i), & \beta_i \in \widetilde{K},\ b_i \in K, \end{array}\right\} \tag{2.1}$$

with $m, n \geq 1$.

2.1 Basic results

In this section we state results which represent the basic principle of root continuity. We give possible bounds for the value $v(f - g)$ which guarantee that the roots of f and g are sufficiently close to each other under a suitable pairing. The following is a result which can be found in [15, Theorem 30.26] and [5, Theorem 2.4.7]. This theorem will be a consequence of Theorem 2.1.3 below (as observed in Remark 2.1.4).

Theorem 2.1.1. *Let f be a separable monic polynomial. Then for every $\varepsilon \in vK$ there exists $\delta \in vK$ such that the following holds:*

If g is a monic polynomial such that $v(f-g) > \delta$, then $\deg g = \deg f$, and for each root α of f there is a root β of g such that $v(\alpha - \beta) \geq \varepsilon$. Moreover, if $\varepsilon > \mathrm{kras}(f)$, then the choice of β is unique and g is separable.

The original version of the above theorem given in [15] has a slightly different formulation. It states that for an arbitrary ε, the choice of β such that $v(\alpha - \beta) \geq \varepsilon$ is unique. However, this is not true for any ε, as can be seen in the following simple example.

Example 2.1.2. Consider $K = \mathbb{Q}$ with the 2-adic valuation v on $\mathbb{Q}$, extended to $\mathbb{Q}[x]$ through the Gauß valuation. Take the polynomials

$$f(x) = g(x) = (x-1)(x+1).$$

Choose $\varepsilon = \mathrm{kras}(f) = 1$. We have $v(f-g) > \delta$ for any $\delta \in vK$, but for the root $\alpha := 1$ of f, both roots $\beta_1 := 1$ and $\beta_2 := -1$ of g satisfy:

$$v(\alpha - \beta_1) \geq \varepsilon \quad \text{and} \quad v(\alpha - \beta_2) \geq \varepsilon.$$

Thus, the pairing between the roots of f and g is not unique.

We will now prove a theorem which provides more detailed information than Theorem 2.1.1. For $f \in K[x]$ as in (2.1), we define:

$$\gamma(f) := \min_{1 \leq i \leq n} v\alpha_i, \quad \gamma^*(f) := \min\{\gamma(f), 0\}. \tag{2.2}$$

Recall from the Introduction that

$$\mathrm{kras}(f) = \{v(\alpha - \alpha') \mid \alpha \neq \alpha' \text{ are roots of } f\}$$

if f has at least two distinct roots, and $\mathrm{kras}(f) = v\alpha$ if f has only one root α.

Theorem 2.1.3. *Take monic polynomials $f, g \in K[x]$ and set*

$$\varepsilon := \frac{v(f-g)}{n} + \gamma^*(f).$$

If $\varepsilon > 0$, then the following assertions hold:

(a) $\deg g = \deg f$,

(b) for each root β of g there is a root α of f such that $v(\alpha - \beta) \geq \varepsilon$,

(c) for each root α of f there is a root β of g such that $v(\alpha - \beta) \geq \varepsilon$,

(d) $\gamma^*(f) = \gamma^*(g)$*, and if* $\varepsilon > \gamma(f)$*, then* $\gamma(f) = \gamma(g)$*.*

Moreover, if f *is separable and* $\varepsilon > \operatorname{kras}(f)$*, then:*

(e) *the root* α *in assertion* (b) *and the root* β *in assertion* (c) *are uniquely determined,*

(f) g *is separable,*

(g) *for every root* α *of* f *the ultrametric ball* $B_\varepsilon(\alpha)$ *contains precisely one root of* f *and precisely one root of* g*,*

(h) $\operatorname{kras}(f) = \operatorname{kras}(g)$*.*

Proof. Let f and g be given by (2.1). Since $\varepsilon > 0$, we have that

$$v(f-g) = n\varepsilon - n\gamma^*(f) > -n\gamma^*(f) \geq 0. \tag{2.3}$$

Suppose that $\deg g \neq \deg f$. Then $g - f$ is a monic polynomial, thus $v(f - g) \leq 0$, which contradicts (2.3). Therefore, we must have $\deg g = \deg f$ and we have proved assertion (a).

To prove assertion (b), suppose that there exists a root β of g such that $v(\alpha_i - \beta) < \varepsilon$ for every i. Then

$$vf(\beta) = \sum_{i=1}^{n} v(\beta - \alpha_i) < n\varepsilon. \tag{2.4}$$

Assume first that $v\beta \geq 0$. By Lemma 1.6.1 with $i = 0$ applied to $f - g$ we have that

$$n\varepsilon \leq n\varepsilon - n\gamma^*(f) = v(f-g) \leq v(f(\beta) - g(\beta)) = vf(\beta),$$

which contradicts (2.4).

Assume now that $v\beta < 0$. Again by Lemma 1.6.1, we obtain that $vf(\beta) \geq v(f-g) + nv\beta$, so

$$n\varepsilon - n\gamma^*(f) = v(f-g) \leq vf(\beta) - nv\beta < n\varepsilon - nv\beta. \tag{2.5}$$

This implies that $v\beta < \gamma^*(f)$. But this means that for all i we have that $v\beta < v\alpha_i$ and therefore $v(\beta - \alpha_i) = v\beta$. Hence,

$$vf(\beta) = \sum_{i=1}^{n} v(\beta - \alpha_i) = nv\beta.$$

Combining this with (2.5), we obtain that

$$v(f-g) \leq vf(\beta) - nv\beta = 0,$$

which contradicts (2.3). This shows that assertion (b) holds.

Now we will show with the same methods that for every root α of f there exists a root β of g such that $v(\alpha-\beta) > \varepsilon$. Suppose there exists a root α of f such that for every root β of g we have that $v(\alpha-\beta) < \varepsilon$. Then

$$vg(\alpha) = \sum_{j=1}^{n} v(\alpha-\beta) < n\varepsilon. \tag{2.6}$$

If $v\alpha \geq 0$, then as before we apply Lemma 1.6.1 for $i = 0$ to $f - g$ to obtain that $n\varepsilon \leq v(f-g) \leq v(f(\alpha)-g(\alpha)) = vg(\alpha)$, which contradicts (2.6).

If $v\alpha < 0$, then by Lemma 1.6.1, $vg(\alpha) - nv\alpha \geq v(f-g)$. Thus,

$$n\varepsilon - n\gamma^*(f) = v(f-g) \leq vg(\alpha) - nv\alpha < n\varepsilon - nv\alpha,$$

whence $v\alpha < \gamma^*(f) \leq \gamma(f) = \min_i v\alpha_i \leq v\alpha$, a contradiction. This shows that assertion (c) holds.

To prove assertion (d), assume first that $\varepsilon > \gamma(f)$. Take k, ℓ such that $v\alpha_\ell = \gamma(f)$ and $v\beta_k = \gamma(g)$. By part (c), there exists a root β of g such that

$$v(\alpha_\ell - \beta) \geq \varepsilon > \gamma(f) = v\alpha_\ell.$$

Thus $\gamma(f) = v\alpha_\ell = v\beta \geq \gamma(g)$. By part (b), there exists a root α of f such that $v(\alpha - \beta_k) \geq \varepsilon$. Since $\varepsilon > \gamma(f)$ and $v\alpha \geq \gamma(f)$, we have that

$$\gamma(g) = v\beta_k \geq \min\{v(\alpha-\beta_k), v\alpha\} \geq \gamma(f).$$

This shows that $\gamma(f) = \gamma(g)$.

It remains to prove that $\gamma^*(f) = \gamma^*(g)$ always holds. If $\varepsilon > \gamma(f)$, then this is a consequence of the equality $\gamma(f) = \gamma(g)$. Now assume that $\varepsilon \leq \gamma(f)$; this implies that $\gamma(f) > 0$. Take β_k as above and use part (b) to find a root α of f such that $v(\alpha - \beta_k) \geq \varepsilon > 0$. Since $v\alpha \geq \gamma(f) > 0$, we obtain that $\gamma(g) = v\beta_k > 0$. Consequently, $\gamma^*(f) = 0 = \gamma^*(g)$.

Assume now that f is separable and $\varepsilon > \mathrm{kras}(f)$. We know by assertion (b) that for every root β of g there is a root α of f such that $v(\beta - \alpha) \geq \varepsilon$. Suppose that for some $i \neq j$ we have that $v(\beta - \alpha_i) \geq \varepsilon$ and $v(\beta - \alpha_j) \geq \varepsilon$. It follows that $v(\alpha_i - \alpha_j) \geq \varepsilon > \mathrm{kras}(f) = \max_{i \neq j} v(\alpha_i - \alpha_j)$, which is a contradiction. This shows that α is uniquely determined, and we also see that the ultrametric balls $B_\varepsilon(\alpha_i)$, $1 \le i \le \deg f$, are pairwise disjoint. By assertion (c), each of the $\deg f$ many balls $B_\varepsilon(\alpha_i)$ contains at least one root of g. As $\deg f = \deg g$, this root is uniquely determined and g is separable. This proves parts (e), (f) and (g).

Take any two distinct roots β_k and β_ℓ of g. Let α_i and α_j be the distinct roots of f such that $\beta_k \in B_\varepsilon(\alpha_i)$ and $\beta_\ell \in B_\varepsilon(\alpha_j)$. Then $v(\alpha_i - \beta_k) \ge \varepsilon > v(\alpha_i - \alpha_j)$ and $v(\alpha_j - \beta_\ell) \ge \varepsilon > v(\alpha_i - \alpha_j)$, whence

$$v(\beta_k - \beta_\ell) = \min\{v(\alpha_i - \beta_k), v(\alpha_j - \beta_\ell), v(\alpha_i - \alpha_j)\} = v(\alpha_i - \alpha_j).$$

Therefore, every value $v(\beta_k - \beta_\ell)$ appears among the values $v(\alpha_i - \alpha_j)$. Since for any distinct α_i and α_j we can also find β_k and β_ℓ as above, we see that also every value $v(\alpha_i - \alpha_j)$ appears among the values $v(\beta_k - \beta_\ell)$. This implies that $\mathrm{kras}(f) = \mathrm{kras}(g)$ and concludes the proof of our theorem. □

Remark 2.1.4. Observe that Theorem 2.1.3 implies Theorem 2.1.1. Indeed, if we choose any $\varepsilon > 0$, then by Theorem 2.1.3 for every polynomial g such that

$$v(f - g) > \delta := n\varepsilon - n\gamma^*(f),$$

the claims of Theorem 2.1.1 are satisfied.

The three following lemmas can be found in varying forms in sources such as [1] (3.4.1, Proposition 3.4.1/1), [9] (Lemma 5.8, Lemma 5.9) and [14] (Lemma 1-3, Lemma 1-4). They present useful observations which will allow us to prove more refined results on the continuity of roots throughout the dissertation.

Lemma 2.1.5. *If α is a root of a monic polynomial $f \in K[x]$, then $v\alpha \ge vf$. In particular, $\gamma^*(f) \ge vf$.*

Proof. Since f is monic, we have that $vf \le 0$, so the first claim is satisfied if $v\alpha \ge 0$. Thus we may assume that $v\alpha < 0$. Write f as in (2.1) with $a_n = 1$. Since

$$nv\alpha = v(\alpha^n) = v\left(\sum_{0\le i<n} a_i\alpha^i\right) \ge \min_{0\le i<n}\{va_i + iv\alpha\},$$

we have that

$$v\alpha \ge \min_{0\le i<n}\{va_i + (i+1-n)v\alpha\} \ge \min_{0\le i<n} va_i \ge vf.$$

□

In view of the above lemma, we can replace the term $\gamma^*(f)$ by vf in the definition of ε in Theorem 2.1.3. This proves useful in case we have no immediate knowledge of the roots of f. Indeed, vf is straightforward to obtain. The value $\gamma^*(f)$, on the other hand, requires computing the slopes of the Newton Polygon NP_f (see Section 3.2 for more details).

Lemma 2.1.6. *Let $f, g \in K[x]$ be polynomials of degree $n \geq 1$. Assume that f is monic and take a root α of f. Then $vg(\alpha) \geq v(f-g) + nvf$.*

Proof. Write $f(x)$, $g(x)$ as in (2.1) with $m = n$, and choose a root α of f. We apply Lemma 1.6.1 for $i = 0$ together with Lemma 2.1.5 and the fact that $vf \leq 0$ to obtain that $vg(\alpha) \geq v(f-g) + \min\{0, nv\alpha\} \geq v(f-g) + nvf$. □

The following lemma is sometimes (e.g. in [1] and [14]) cited as a separate result on the continuity of roots. It is a generalization of one of the results given in Theorem 2.1.1, since we don't require the polynomial g to be monic. This lemma will be employed, directly or indirectly, to prove a number of results (see e.g. Theorem 2.1.8, Theorem 2.2.2 and Theorem 2.3.2).

Lemma 2.1.7. *Let $f, g \in K[x]$ be polynomials of degree $n \geq 1$, assume that f is monic and let α be a root of f. If g is monic or $v(f-g) > 0$, then there exists a root β of g such that*

$$v(\beta - \alpha) \geq vf + \frac{v(f-g)}{n}.$$

Proof. Write f and g as in (2.1) with $m = n$. We first claim that $vb_n = 0$. This is true if g is monic. If $v(f-g) > 0$, then

$$0 < v(f-g) = \min_i\{v(a_i - b_i)\} \leq v(1 - b_n),$$

which also implies that $vb_n = 0$. Suppose that for every root β of g we have:

$$v(\beta - \alpha) < vf + \frac{v(f-g)}{n}.$$

Since $vb_n = 0$, we thus obtain that

$$vg(\alpha) = \sum_{i=1}^{n} v(\beta_i - \alpha) \leq n \cdot \max_{1 \leq i \leq n} v(\beta_i - \alpha) < nvf + v(f-g),$$

which contradicts Lemma 2.1.6. □

The following theorem is a direct application of Lemma 2.1.7 and Theorem 2.1.3. We employ the results and methods which were already introduced, in order to formulate a root continuity theorem which does not require the polynomials in question to be monic. We are, however, assuming that they are of equal degree. Another price to pay for the generalization is that the bound in the following theorem can be worse than the one in Theorem 2.1.3.

Theorem 2.1.8. *Let $f \in K[x]$ be as in (2.1) and take $\varepsilon > 0$. If $g \in K[x]$ is a polynomial of degree n such that*

$$v(f - g) \geq n\varepsilon - nvf + (n+1)va_n, \tag{2.7}$$

then assertions (b)–(d) of Theorem 2.1.3 hold. Moreover, if f is separable and $\varepsilon > \mathrm{kras}(f)$, then also assertions (e)–(h) of Theorem 2.1.3 hold.

Proof. Observe that Equation (2.7) is equivalent to:

$$v(a_n^{-1}f - a_n^{-1}g) \geq n\varepsilon - nv(a_n^{-1}f).$$

We will work with $\hat{g} := a_n^{-1}g$ and with the monic polynomial $\hat{f} := a_n^{-1}f$. Both polynomials have the same roots as g and f respectively. Our assumption can now be written as:

$$v(\hat{f} - \hat{g}) \geq n\varepsilon - nv\hat{f} \geq n\varepsilon > 0.$$

Fix any root α of $\hat{f}$. From Lemma 2.1.7 we infer that there exists a root β of $\hat{g}$ such that

$$v(\alpha - \beta) \geq v\hat{f} + \frac{v(\hat{f} - \hat{g})}{n} \geq v\hat{f} + \varepsilon - v\hat{f} = \varepsilon.$$

This proves assertion (c) of Theorem 2.1.3.

Observe that $v(1 - a_n^{-1}b_n) \geq v(\hat{f} - \hat{g}) > 0$. This implies that $v(a_n^{-1}b_n) = 0$, and so $va_n = vb_n$. Moreover, we see that $v(f - g) > va_n$, which together with $vf \leq va_n$ implies that $vf = vg$. Working with $b_n^{-1}f$ and the monic polynomial $b_n^{-1}g$, our assumption now states that:

$$v(b_n^{-1}f - b_n^{-1}g) \geq n\varepsilon - nv(b_n^{-1}g).$$

Thus we can repeat the above method to prove part (b) of Theorem 2.1.3.

To prove the further assertions, we observe that the arguments for assertions (d)–(h) in the proof of Theorem 2.1.3 do not use the assumption that the polynomials in question are monic. Thus we can employ the now proved assertions (b) and (c) and repeat the rest of the proof of Theorem 2.1.3. □

2.2 Convergent nets of polynomials

A different approach to root continuity can be found in [14]. Instead of looking at a polynomial which is 'close' to a given polynomial, we consider convergent nets of polynomials and study the behavior of their roots.

A *directed* set $(I, \leq)$ is a partially ordered set such that for all $i, j \in I$ there is $k \in I$ such that $k \geq i$ and $k \geq j$. We call $J \subseteq I$ *cofinal* in I if for every $i \in I$ there is $j \in J$ such that $j \geq i$. Note that a cofinal subset of a directed set is itself directed. Moreover, if I_1 is cofinal in I_2 and I_2 is cofinal in I_3, then I_1 is cofinal in I_3.

A *net* in a set X is a function $\varphi : I \to X$, where I is a directed set; we will denote it by $(x_i)_{i\in I}$. For $Y \subseteq X$, we say that $(x_i)_{i\in I}$ is *ultimately in* Y if there is some $i_0 \in I$ such that $x_i \in Y$ for each $i \in I$ with $i \geq i_0$.

Now assume that X is a topological space. An element $x \in X$ is a *limit of the net* $(x_i)_{i\in I}$ if for every open neighborhood U_x of x, $(x_i)_{i\in I}$ is ultimately in U_x. This fact shall be written as follows: $(x_i)_{i\in I} \to x$. In this case we will say that the net $(x_i)_{i\in I}$ is *convergent* and that it *converges* to x.

Finally, let I and J be directed sets. We say that $(x_j)_{j\in J}$ is a *subnet of* $(x_i)_{i\in I}$ if J is a cofinal subset of I.

Lemma 2.2.1. *Let I be a directed set such that $I = I_1 \cup \ldots \cup I_n$. Then there exists $k \in \{1, \ldots, n\}$ such that I_k is cofinal in I.*

In particular, if under these assumptions $(x_i)_{i\in I}$ is a net, then $(x_i)_{i\in I_k}$ is a subnet.

Proof. If I is finite, it must have a largest element, which then must be in some I_k, which proves the claim. Assume that I is infinite. Suppose that for every $k \in \{1, \ldots, n\}$ there exists j_k such that $j_k > I_k$. Since I is directed, there exists $j \in I$ such that $j \geq j_k$ for every k. But then $j > I_k$ for every k and thus $j > I$, a contradiction. □

A particular case of convergence that will be considered in this dissertation is given by the topology induced by a valuation v on a valued field or ring X. In this setting, we have that $(x_i)_{i\in I} \to x$ if for all $r \in vX$ there is some $i_0 \in I$ such that $v(x_i - x) > r$ for each $i \in I$ with $i \geq i_0$.

The following result can be found in [14, Lemma 1-6].

Theorem 2.2.2. *Let (K, v) be a valued field and let $(I, \leq)$ be a directed set. Consider a net $(f_i)_{i\in I}$ of monic polynomials in $K[x]$ of degree n. Moreover, let $f \in K[x]$ be the limit of $(f_i)_{i\in I}$ with respect to the valuation v, and for any $i \in I$ choose a root β_i of f_i. Then there exists a root α of f and a cofinal subset $J \subseteq I$ such that $(\beta_j)_{j\in J} \to \alpha$.*

Proof. Choose $(f_i)_{i\in I}$, $\beta_i \in \widetilde{K}$ and $f \in K[x]$ as in the theorem. Note that since $(f_i)_{i\in I} \to f$, the set $I_0 := \{i \in I \mid v(f_i - f) > 0\}$ is cofinal in I. If we find J cofinal in I_0 which satisfies our claim, then J will also be cofinal in I. We can therefore assume without loss of generality that $v(f_i - f) > 0$ for all $i \in I$. Since f is a limit of monic polynomials of degree n, it is itself a monic polynomial of degree n. This fact combined with $v(f_i - f) > 0$ shows that $vf = vf_i$ for all $i \in I$.

Let $\alpha_1, \ldots, \alpha_n \in \widetilde{K}$ be all the (not necessarily distinct) roots of f. For each $k \in \{1, \ldots, n\}$ define:

$$J(\alpha_k) := \left\{ i \in I \mid v(\alpha_k - \beta_i) \geq \frac{v(f - f_i)}{n} + vf \right\}.$$

By Lemma 2.1.7 we have that for each $i \in I$ there exists $k \in \{1, \ldots, n\}$ such that $i \in J(\alpha_k)$, that is:

$$I = J(\alpha_1) \cup \ldots \cup J(\alpha_n).$$

By Lemma 2.2.1 there exists a root α of f such that $J(\alpha)$ is cofinal in I. Set $J := J(\alpha)$. Then $(f_j)_{j\in J}$ is a net convergent to f, that is, for all $r \in vK$ there is some $j_0 \in J$ such that $v(f - f_j) > r$ for each $j \in J$ with $j \geq j_0$. Fix any element $\varepsilon \in vK$. We wish to show that, ultimately, $v(\alpha - \beta_j) > \varepsilon$. We know that there is some $j_1 \geq j_0$ such that $v(f - f_j) \geq n\varepsilon - nvf$ for each $j \in J$ with $j \geq j_1$. Thus for all $j \in J$ such that $j \geq j_1$ the following holds:

$$v(\alpha - \beta_j) \geq \frac{v(f - f_j)}{n} + vf \geq \frac{n\varepsilon - nvf}{n} + vf = \varepsilon.$$

□

The following corollary can be immediately deduced from Theorem 2.2.2. However, it can also be proved by a direct application of Lemma 2.1.7, as observed in [1, Sect. 3.4, Corollary 2].

Corollary 2.2.3. *Consider a sequence $(f_i)_{i\in\mathbb{N}}$ of monic polynomials in $K[x]$ of degree n such that $(f_i)_{i\in\mathbb{N}} \longrightarrow f \in K[x]$. For any $i \in \mathbb{N}$ choose a root β_i of f_i. Then the sequence $(\beta_i)_{i\in\mathbb{N}}$ contains a subsequence which converges to a root of f.*

Example 2.2.4. Since the choice of the respective roots β_i is arbitrary, we only have convergence up to a cofinal subset in Theorem 2.2.2 and a subsequence in Corollary 2.2.3. Indeed, take $I := \mathbb{N}$. Consider the polynomials $f_i(x) = f(x) = (x - \alpha_1)(x - \alpha_2)$ with $\alpha_1 \neq \alpha_2$, and set $\beta_i = \alpha_1$ for even i and $\beta_i = \alpha_2$ for odd i. Then $(\beta_i)_{i\in I} \not\to \alpha_1$ and $(\beta_i)_{i\in I} \not\to \alpha_2$.

From Theorem 2.2.2 we know that if $(f_i)_{i\in I} \to f$, then any net of roots β_i of f_i contains a subnet convergent to some root α of f. By directly applying Theorem 2.1.3, we can find a converse result: any root α of f is obtained as a limit of a suitable choice of $(\beta_i)_{i\in I}$.

Theorem 2.2.5. *Let $(I, \leq)$ be a directed set and assume that $(f_i)_{i\in I}$ is a net of monic polynomials in $K[x]$ of degree n with limit $f \in K[x]$. Choose a root α of f. Then there exists a net $(\beta_i)_{i\in I}$ of elements of $\widetilde{K}$ such that β_i is a root of f_i for each $i \in I$, and $(\beta_i)_{i\in I} \to \alpha$.*

Proof. For each $i \in I$ we choose a root β_i of f_i such that

$$v(\alpha - \beta_i) \geq v(\alpha - \beta_i')$$

for every root β_i' of f_i. Fix any $\varepsilon \in vK$. Then there exists $i_0 \in I$ such that for all $i \in I$, $i \geq i_0$, we have that:

$$v(f - f_i) > n\varepsilon - n\gamma^*(f).$$

By Theorem 2.1.3 (see also Remark 2.1.4) we have that there exists a root β_i' of f_i such that $v(\alpha - \beta_i') \geq \varepsilon$. By our choice of β_i we also have that

$$v(\alpha - \beta_i) \geq \varepsilon,$$

thus $(\beta_i)_{i\in I} \to \alpha$. $\square$

Remark 2.2.6. In Chapter 4 we will state Theorem 4.2.5 which will allow us to refine the result above. We assume that $(f_i)_{i\in I} \to f$ and we choose a root α of f of multiplicity t. Then for every $\varepsilon \geq \mathrm{kras}(f)$ there exists $i_0 \in I$ such that for every $i \geq i_0$, each of the polynomials f_i has precisely t many roots in the open ultrametric ball $B_\varepsilon^\circ(\alpha)$.

2.3 Induction on the degree of the polynomial

In this section we present a theorem whose essential feature is that its proof employs induction on the degree of the polynomials. It serves as a demonstration of what can be achieved through this method. The bound for the value $v(f - g)$ will be larger than the ones in Chapter 4, which makes it a less optimal method than those which will be presented later in the dissertation.

The method can be described as follows: first set

$$f_1 := f \quad \text{and} \quad g_1 := g$$

and choose any root α_1 of f. Then use Lemma 2.1.7 to find a root β_1 of g such that $v(\alpha_1 - \beta_1) > \varepsilon$. Then set

$$f_2 := \frac{f_1}{x - \alpha_1} \quad \text{and} \quad g_2 := \frac{g_1}{x - \beta_1}$$

to repeat the procedure and continue the process until we arrive at linear polynomials.

To prove the main theorem, we first need the following lemma.

Lemma 2.3.1. *Take $f, g \in K[x]$ with f monic and let α be a root of f. Assume that $v(f - g) > 0$ and that β is a root of g such that*

$$v(\alpha - \beta) \geq vf + \frac{v(f-g)}{n}.$$

Then

$$v\left(\frac{f(x)}{x-\alpha} - \frac{g(x)}{x-\beta}\right) \geq 2vf + \frac{v(f-g)}{n}.$$

Proof. Since $(x-\alpha)(x-\beta)$ is monic, we have that $v((x-\alpha)(x-\beta)) \leq 0$. Therefore, we obtain that

$$\begin{aligned} v\left(\frac{f(x)}{x-\alpha} - \frac{g(x)}{x-\beta}\right) &= v(f(x)(x-\beta) - g(x)(x-\alpha)) - v((x-\alpha)(x-\beta)) \\ &\geq v(f(x)(x-\beta) - g(x)(x-\alpha)) \\ &\geq \min\left\{v\left((f(x)-g(x))x\right), v\left(f(x)\beta - g(x)\alpha\right)\right\} \\ &= \min\left\{v(f-g), v\left(f(x)\beta - g(x)\alpha\right)\right\}. \end{aligned}$$

We wish to find a lower bound for $v(f(x)\beta - g(x)\alpha)$. We use the assumption of the lemma and the facts that $vf \leq 0$ and $vf \leq v\alpha$ (since f is monic) to obtain:

$$\begin{aligned} v\left(f(x)\beta - g(x)\alpha\right) &= v\left(f(x)(\beta-\alpha) + (f(x)-g(x))\alpha\right) \\ &\geq \min\left\{vf + v(\beta-\alpha), v(f-g) + v\alpha\right\} \\ &\geq \min\left\{2vf + \frac{v(f-g)}{n}, v(f-g) + vf\right\} \\ &= 2vf + \frac{v(f-g)}{n}. \end{aligned}$$

Going back to the initial inequality, we obtain that

$$v\left(\frac{f(x)}{x-\alpha} - \frac{g(x)}{x-\beta}\right) \geq \min\left\{v(f-g), 2vf + \frac{v(f-g)}{n}\right\} = 2vf + \frac{v(f-g)}{n}.$$

□

Theorem 2.3.2. *Take polynomials f, g as in (2.1) and $\varepsilon > 0$. Assume that*

$$v(f-g) > n!\,\varepsilon - (n+1)!\,(vf - va_n) + va_n. \tag{2.8}$$

Then there is an enumeration of the roots of g such that $v(\alpha_i - \beta_i) \geq \varepsilon$ for $1 \leq i \leq n$.

Proof. Condition (2.8) can be written in a simpler way, with f replaced by the monic polynomial $a_n^{-1} f$:

$$v(a_n^{-1} f - a_n^{-1} g) > n!\varepsilon - (n+1)!(v a_n^{-1} f).$$

Since f and $a_n^{-1} f$ have the same roots and the same is true for g and $a_n^{-1} g$, we may assume that f is monic. In this case, $va_n = 0$ and $vf \leq 0$.

We will proceed by reverse induction on the degree n of f as long as it is larger than 1. We set $f_1 := f$ and $g_1 := g$ and choose any root α_1 of f. We use Lemma 2.1.7 to find a root β_1 of g such that

$$\begin{aligned} v(\alpha_1 - \beta_1) &\geq vf_1 + \frac{v(f_1 - g_1)}{n} > vf_1 + \frac{n!\,\varepsilon - (n+1)!\,vf_1}{n} \\ &= vf + (n-1)!\,\varepsilon - (n+1)(n-1)!\,vf \\ &= (n-1)!\,\varepsilon - n!\,vf - ((n-1)! - 1)vf \geq \varepsilon, \end{aligned}$$

where we have used that $n - 1 \geq 1$ and $vf_1 = vf \leq 0$.

Now we assume that $i < n$ and that for $1 \leq j \leq i$, we have already found roots α_j, β_j such that $v(\alpha_j - \beta_j) \geq \varepsilon$, and polynomials f_j, g_j such that

$$\deg f_j = \deg g_j = n - j + 1,$$

as well as $vf_j \geq vf$ and

$$v(f_j - g_j) \geq (n-j+1)!\,\varepsilon - (n-j+2)!\,vf. \tag{2.9}$$

We define

$$f_{i+1} := \frac{f_i}{x - \alpha_i} \quad \text{and} \quad g_{i+1} := \frac{g_i}{x - \beta_i}.$$

Observe that

$$\deg f_{i+1} = \deg g_{i+1} = n - (i+1) + 1$$

and that $vf_{i+1} \geq vf_i \geq vf$ because $v(x - \alpha_i) \leq 0$. By Lemma 2.3.1 we have:

$$\begin{aligned} v(f_{i+1} - g_{i+1}) &\geq 2vf_i + \frac{v(f_i - g_i)}{n-i+1} \\ &\geq 2vf + \frac{(n-i+1)!\,\varepsilon - (n-i+2)!\,vf}{n-i+1} \\ &= 2vf + (n-i)!\,\varepsilon - (n-i+2)(n-i)!\,vf \\ &= (n-i)!\,\varepsilon - (n-i+1)!\,vf - ((n-i)! - 2)vf. \end{aligned}$$

If $i+1<n$, then

$$((n-i)!\,-2)\geq 0$$

and we obtain that (2.9) also holds for $j=i+1$. If $i+1=n$, then

$$-(n-i+1)!-((n-i)!-2)=-1,$$

thus

$$v(f_n-g_n)\geq \varepsilon - vf. \tag{2.10}$$

Assume that $i+1<n$. Then $\deg f_{i+1}=\deg g_{i+1}>1$ and we have to continue our induction. We choose a root α_{i+1} of f_{i+1}. Then by Lemma 2.1.7 there is a root β_{i+1} of g_{i+1} such that

$$\begin{aligned} v(\alpha_{i+1}-\beta_{i+1}) &\geq vf_{i+1}+\frac{v(f_{i+1}-g_{i+1})}{n-i} \\ &\geq vf+(n-i-1)!\,\varepsilon-(n-i+1)(n-i-1)!\,vf\geq\varepsilon, \end{aligned}$$

where we use that $n-i-1\geq 1$. This completes our induction step.

Finally, we deal with the case of $i+1=n$. Then both f_{i+1} and g_{i+1} are linear polynomials, say, $x-\alpha$ and $b_n(x-\beta)$. We set $\alpha_n:=\alpha$. In view of (2.10), Lemma 2.1.7 shows the existence of a root β_n of g_n, which consequently must be equal to β, such that

$$v(\alpha_n-\beta_n)\geq vf_n+v(f_n-g_n)\geq\varepsilon.$$

This completes the proof of our theorem. □

Chapter 3

The Newton Polygon

3.1 Lines, segments and Newton Polygons of certain finite sets

Let $(\Gamma_0, +)$ be an ordered Abelian group, and let I be an ordered set such that the order type of I equals the principal rank of Γ_0. From Section 1.5 we know that Γ_0 is isomorphic to a subgroup of the Hahn product $\Gamma := \mathbf{H}_{i\in I}\mathbb{R}$ endowed with the lexicographic order. We will identify Γ_0 with its image in Γ and work with the group Γ in place of Γ_0. In this manner, Γ becomes a vector space over $\mathbb{R}$. For our purposes, however, it is sufficient to have a vector space over $\mathbb{Q}$. This is achieved by considering the divisible hull of Γ_0.

In what follows, we will be working with points in the Cartesian product $\mathbb{R} \times \Gamma$, keeping in mind that all the notions and facts can also be formulated when working with points in $\mathbb{Q} \times \widetilde{\Gamma_0}$.

We will commonly use terms "left/right" to signify locations of points with respect to the first coordinate, and "above/below" when speaking about locations of points with respect to the second coordinate.

Consider two points $p_1, p_2 \in \mathbb{N}_0 \times \Gamma$, with $p_i = (k_i, \gamma_i)$, $i = 1, 2$, and $k_1 \neq k_2$. *The line going through* p_1 *and* p_2 is the set

$$L(p_1, p_2) := \{(x, \gamma) \in \mathbb{R} \times \Gamma \mid (k_2 - k_1)(\gamma - \gamma_1) = (\gamma_2 - \gamma_1)(x - k_1)\}.$$

Note that $L(p_1, p_2) = L(p_2, p_1)$. Moreover, since $k_1 \neq k_2$, we may define the line using a linear function $\ell : \mathbb{R} \to \Gamma$ as follows:

$$\ell(x) = \left(\frac{\gamma_2 - \gamma_1}{k_2 - k_1}\right) x + \left(\gamma_1 + \frac{\gamma_1 - \gamma_2}{k_2 - k_1} k_1\right).$$

Then

$$L(p_1, p_2) = \{(x, \ell(x)) \in \mathbb{R} \times \Gamma \mid x \in \mathbb{R}\}.$$

The slope of $L(p_1, p_2)$ is the slope of the corresponding function ℓ, that is,

$$s(\ell) := \frac{\gamma_2 - \gamma_1}{k_2 - k_1}.$$

Assume that $k_1 < k_2$. *The segment connecting* p_1 *and* p_2 is the set

$$[p_1, p_2] := [p_2, p_1] := \{(x, \ell(x)) \in \mathbb{R} \times \Gamma \mid k_1 \leq x \leq k_2\}.$$

The slope of $[p_1, p_2]$ is the slope of the corresponding line $L(p_1, p_2)$, which we will denote by $s[p_1, p_2]$.

From now on until the end of this section we consider a finite subset A of points $p_i := (k_i, \gamma_i) \in \mathbb{N}_0 \times \Gamma$, $1 \leq i \leq r$ such that $k_i < k_{i+1}$ for $1 < i < r$. We will construct a subset $\phi(A) \subseteq A$ inductively in the following manner.

In the first step, take $i_1 := 1$ and $\phi_1(A) := \{p_{i_1}\}$. If $r = 1$, then we set $\phi(A) := \phi_1(A)$ and finish the construction. Otherwise, we continue our induction as follows.

Assume that we have already constructed the sets $\phi_1(A), \ldots, \phi_j(A)$ and indices $i_1, \ldots, i_j$ for $i_j < r$. We define

$$i_{j+1} := \max\{i_j < i \leq r \mid s[p_{i_j}, p_i] \leq s[p_{i_j}, p_j] \text{ for all } j \in \{i_j + 1, \ldots, r\}\}.$$

In other words, we look at all the points p_i, $i_j < i \leq r$, such that the slope of the segment $[p_{i_j}, p_i]$ is minimal, then among all such points we choose $p_{i_{j+1}}$ to be the one with the largest index. We then set $\phi_{j+1}(A) := \phi_j(A) \cup \{p_{i_{j+1}}\}$. If $i_{j+1} < r$, then we continue the induction.

Since the set A is finite, we will end up at $i_s = r$ for some $s \in \mathbb{N}$. In this manner we obtain the set $\phi(A) := \phi_s(A) = \{p_{i_1}, \ldots, p_{i_s}\}$.

Define $\ell_j : \mathbb{R} \to \Gamma$ to be the linear function corresponding to the line going through p_{i_j} and $p_{i_{j+1}}$, $1 \leq j < s$. Observe that $\ell_j(k_{i_{j+1}}) = \ell_{j+1}(k_{i_{j+1}})$. Denote by ∞ an element greater than every element in Γ and define the function $\mathrm{NP}_A : \mathbb{R} \to \Gamma \cup \{\infty\}$ as follows:

$$\mathrm{NP}_A(x) := \begin{cases} \infty, & \text{if } x < k_1 \\ \ell_j(x), & \text{if } k_{i_j} \leq x \leq k_{i_{j+1}} \text{ for some } 1 \leq j < s, \\ \infty, & \text{if } x > k_r. \end{cases}$$

We will call this function *the Newton Polygon of the set A*. *The graph* of this function is the set

$$\mathcal{G}(\mathrm{NP}_A, [k_1, k_r]) := \{(x, \mathrm{NP}_A(x)) \in \mathbb{R} \times \Gamma \mid k_1 \leq x \leq k_r\}.$$

In general, assume that there exist linear functions $\delta_1, \ldots, \delta_t$ and elements $x_1 < \ldots < x_{t+1} \in \mathbb{R}$ such that $\delta_j(x_{j+1}) = \delta_{j+1}(x_{j+1})$ for $1 \le j < t$. If a function f is given by

$$f(x) = \delta_j(x) \text{ if } x_j \le x \le x_{j+1} \text{ for some } 1 \le j \le t,$$

then we will say that f is *piecewise linear on* $[x_1, x_{t+1}]$. If f is piecewise linear on every interval contained in its domain, then we say simply say that f is *piecewise linear.* We thus see that NP_A is a piecewise linear function on $[k_1, k_r]$. Directly from the construction of NP_A we also obtain the following proposition.

Proposition 3.1.1. *The Newton Polygon of A has the following properties:*

(a) *the sequence of slopes $s[p_{i_j}, p_{i_{j+1}}]$, $1 \le j < s$, is strictly increasing,*

(b) *each point in A lies on or above $\mathcal{G}(\mathrm{NP}_A, [k_1, k_r])$, that is, $\gamma_i \ge \mathrm{NP}_A(k_i)$ for all $i \in \{1, \ldots, r\}$,*

(c) *$\phi(A)$ is the smallest set among the subsets $B \subseteq A$ such that $\mathrm{NP}_A = \mathrm{NP}_B$.*

Take a convex set $I \subseteq \mathbb{R}$ and $f : I \to \Gamma$. We say that f is *upward convex (on I)* if for any two points $q, q' \in \mathcal{G}(f, I)$ and for every point $(x, \gamma) \in [q, q']$, we have that $\gamma \ge f(x)$.

Proposition 3.1.2. *The restriction of the function NP_A to $[k_1, k_r]$ is an upward convex function.*

Proof. If both q and q' lie on one segment $[p_{i_j}, p_{i_{j+1}}]$, then $[q, q']$ is contained in that segment and the claim is satisfied. Thus assume that this is not the case.

Denote by y and y' the first coordinate of q and q', respectively and assume without loss of generality that $y < y'$. Denote by δ the linear function corresponding to the line going through the points q and q'.

By our assumption, $q \in [p_{i_l}, p_{i_{l+1}}]$ and $q' \in [p_{i_m}, p_{i_{m+1}}]$ for some l, m with $1 \le l < m \le r$. Replacing l with $l + 1$ and m with $m - 1$ if necessary, we may assume without loss of generality that $q \ne p_{i_{l+1}}$ and $q' \ne p_{i_m}$. Note that after said replacement, we will still have that $l < m$, since otherwise q and q' would lie on the same segment, which is a case that we excluded. Then the slopes of the respective linear functions satisfy $s(\ell_l) < s(\delta) < s(\ell_m)$.

Denote by l_0 the largest index in $\{l, \ldots, m\}$ such that $s(\ell_{l_0}) \le s(\delta)$. Then $l_0 + 1$ is the smallest index in $\{l, \ldots, m\}$ such that $s(\ell_{l_0+1}) > s(\delta)$. Take any point $(x, \delta(x)) \in [q, q']$ and assume without loss of generality that $x \neq q$ and $x \neq q'$. We wish to show that $\delta(x) \geq \mathrm{NP}_A(x)$.

Assume first that $y < x \leq k_{i_{l_0+1}}$. If $y < x \leq k_{i_{l+1}}$ (this happens e.g. if $l_0 = l$), then

$$\delta(x) = \delta(y) + (x-y)s(\delta) \geq \delta(y) + (x-y)s(\ell_l) = \mathrm{NP}_A(x).$$

If $k_{i_{l+1}} < x \leq k_{i_{l_0+1}}$, then in particular, $l_0 > l$. Write

$$z := \max\{j \in \{l+1, \ldots, l_0\} \mid k_{i_j} < x\}.$$

Then

$$\begin{aligned}
\delta(x) &= \delta(y) + (x-y)s(\delta) \\
&\geq \delta(y) + (x-y)s(\ell_{l_0}) \\
&\geq \delta(y) + (k_{i_{l+1}} - y)s(\ell_l) + \left(\sum_{j=l+1}^{z-1} (k_{i_{j+1}} - k_{i_j})s(\ell_j)\right) + (x - k_{i_z})s(\ell_z) \\
&= \mathrm{NP}_A(x),
\end{aligned}$$

where the last equality follows from the fact that $(y, \delta(y))$ is a point on NP_A.

We will now prove in an analogous manner that $\delta(x) \geq \mathrm{NP}_A(x)$ if $k_{i_{l_0+1}} < x \leq y'$. If in addition $x \geq k_{i_m}$, then

$$\delta(x) = \delta(y') + (x-y')s(\delta) \geq \delta(y') + (x-y')s(\ell_m) = \mathrm{NP}_A(x).$$

If $k_{i_{l_0+1}} < x < k_{i_m}$, then in particular, $l_0 + 1 < m$. Write

$$z := \min\{j \in \{l_0+2, \ldots, m\} \mid k_{i_j} > x\}.$$

Then

$$\begin{aligned}
\delta(x) &= \delta(y') + (x-y')s(\delta) \\
&\geq \delta(y') + (x-y')s(\ell_{l_0+1}) \\
&\geq \delta(y') + (k_{i_m} - y')s(\ell_m) \\
&+ \left(\sum_{j=z}^{m-1} (k_{i_j} - k_{i_{j+1}})s(\ell_j)\right) + (x - k_{i_z})s(\ell_{z-1}) \\
&= \mathrm{NP}_A(x),
\end{aligned}$$

□

Recall that we are considering the points $p_{i_1}, \dots, p_{i_s} \in \phi(A)$. If $k_{i_1} > 0$, then we will consider an additional point

$$p_0 := (k_0, \gamma_0) := (0, \infty) \in \mathbb{R} \times (\Gamma \cup \{\infty\}).$$

The vertices of NP_A are the points $p_0, p_{i_1}, \dots, p_{i_s}$ if $k_{i_1} > 0$ and the points in $\phi(A)$ otherwise. *The face of* NP_A is any segment of the form $[p_{i_j}, p_{i_{j+1}}]$ and the set

$$[p_0, p_{i_1}] := \{(k_{i_1}, \gamma) \in \mathbb{R} \times \Gamma \mid \gamma \geq \gamma_{i_1}\}.$$

When referring to a set which is either a segment or a set of the above form in a general setting, we will call it *a generalized segment. The slope of a face* $[p_{i_j}, p_{i_{j+1}}]$ is the slope of the corresponding segment $[p_{i_j}, p_{i_{j+1}}]$. *The slope of the face* $[p_0, p_{i_1}]$ is defined to be $-\infty$, where $-\infty$ is a symbol for an element strictly less than every element in Γ. This slope can be intuitively understood as a result of "calculating" the slope as we did in the case of segments:

$$-\infty = \frac{\gamma_{i_1} - \infty}{k_{i_1} - 0}$$

Take $i, j \in \{0, i_1, \dots, i_s\}$, $i < j$. *The length of a face* $[p_i, p_j]$ is $k_j - k_i$. This extended definition which possibly includes an additional point p_0 will be put to use in Section 3.2, where the set A will be defined using coefficients of a given polynomial.

Similarly, we can define faces, slopes and vertices of any piecewise linear function f. That is, *a face of* f is any generalized segment in the graph of f, *the slopes of* f are the slopes of those segments, and *a vertex of* f is any point $(x, f(x))$ such that the slope of f changes at the coordinate x.

We observe that the proof of Proposition 3.1.2 only assumes that the function in question is a piecewise linear function on an interval whose slopes are strictly increasing. Thus we obtain the following fact.

Corollary 3.1.3. *Let f be a piecewise linear function on an interval I and let $[(x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))]$, $1 \leq i \leq t$, be faces of f. Assume that $x_i < x_{i+1}$ for $1 \leq i < t$ and that the slopes of the faces are strictly increasing with i. Then f is an upward convex function.*

The Newton Polygon also has the following property, which leads to it being called the "convex hull" of A in some sources (see e.g. [8]).

Proposition 3.1.4. *The function NP_A is the largest upward convex function on $[k_1, k_r]$ such that every point in A lies on or above its graph.*

In other words, if $f : [k_1, k_r] \to \Gamma$ is any other upward convex function with this property, then $f(y) \leq \mathrm{NP}_A(y)$ for all $y \in [k_1, k_r]$.

If we assume in addition that all the vertices of f are in A, then $f = \mathrm{NP}_A$ on the whole interval $[k_1, k_r]$.

Proof. Suppose that $f(y) > \mathrm{NP}_A(y)$ for some $y \in [k_1, k_r]$. Then the point $(y, f(y))$ lies strictly above some segment $[p_{i_j}, p_{i_{j+1}}]$. However, both p_{i_j} and $p_{i_{j+1}}$ lie on or above the graph of f, that is, $\gamma_{i_j} \geq f(k_{i_j})$ and $\gamma_{i_{j+1}} \geq f(k_{i_{j+1}})$. Thus the point $(y, f(y))$ lies strictly above the segment

$$[(k_{i_j}, f(k_{i_j})), (k_{i_{j+1}}, f(k_{i_{j+1}}))],$$

which means that f is not upward convex.

To prove the final assertion, we take f satisfying the assumptions of the proposition. Observe that if $(x, f(x))$ is a vertex of f, then $f(x) = \mathrm{NP}_A(x)$. Indeed, every point in A lies on or above the graph of NP_A, and we have that $f(x) \leq \mathrm{NP}_A(x)$.

Take any $y \in [k_1, k_r]$. Once we show that $f(y) \geq \mathrm{NP}_A(y)$, we will obtain that $f(y) = \mathrm{NP}_A(y)$ and the proof will be finished. The claim is satisfied if $(y, f(y))$ is a vertex of f, thus assume otherwise. Let $[p, p']$ be the face of f that contains $(y, f(y))$. On the one hand, p and p' are points on the graph of f and thus by the previous assertion of the proposition, p and p' lie on or below NP_A. On the other hand, $p, p' \in A$ by assumption and every point of A lies on or above the graph of NP_A. Therefore, p and p' lie on the graph of NP_A. By the upper convexity of NP_A, $(y, f(y))$ lies on or above the graph of NP_A, thus $f(y) \geq \mathrm{NP}_A(y)$. □

3.2 The Newton Polygon of a polynomial

In this section, we will introduce the notion of Newton Polygons for polynomials over a valued field (K, v). The group Γ considered in the previous section will thus be seen as a group containing vK, as in the definition of a valuation given at the beginning of Section 1.1.1. In that section and in Section 3.1, we have introduced the symbol ∞ for an element strictly greater than every element in Γ. It satisfies

$$\infty = \infty + \infty = \gamma + \infty = \infty + \gamma \quad \text{for all } \gamma \in \Gamma.$$

In Section 3.1 we have also introduced the symbol $-\infty$ for an element strictly less than every element in Γ. For an element $r \in \mathbb{R} \setminus \{0\}$, we define

$$r\infty := \begin{cases} \infty, & \text{if } r > 0 \\ -\infty, & \text{if } r < 0, \end{cases} \qquad r(-\infty) := \begin{cases} -\infty, & \text{if } r > 0 \\ \infty, & \text{if } r < 0. \end{cases}$$

Recall from Chapter 2.1 that the valuation v on $K[x]$ that we are considering is the Gauß valuation.

Consider a monic polynomial $f \in K[x]$ given by (2.1), that is, $a_n = 1$. Observe that the coefficients are symmetric functions in the roots:

$$a_{n-k} = s_k(\alpha_1, \ldots, \alpha_n) := (-1)^k \sum_{1 \le i_1 < \ldots < i_k \le n} \alpha_{i_1} \cdot \ldots \cdot \alpha_{ik}.$$

For a (not necessarily monic) polynomial $f \in K[x]$ given by (2.1) we will commonly use the following notation.

Notation 3.2.1. We enumerate the roots α_i of f so that

$$v\alpha_1 \le v\alpha_2 \le \ldots \le v\alpha_n.$$

We denote by $\gamma_1, \ldots, \gamma_s$ the distinct values of roots of f, with $\gamma_i < \gamma_{i+1}$ for $1 \le i < s$. Moreover, we take γ_0 to be an arbitrary value in vK such that $\gamma_0 < \gamma_1$. We define $k_i \in \mathbb{N}_0$ to be the number of roots of f of value strictly greater than γ_i, and $j_i \in \mathbb{N}_0$ to be the number of roots of f of value at most γ_i, $0 \le i \le s$.

We then have that

$$0 = j_0 < j_1 < \ldots < j_{s-1} < j_s = n.$$

Moreover,

$$v\alpha_{j_{\ell-1}+1} = v\alpha_j = v\alpha_{j_\ell} = \gamma_\ell$$

for $1 \le \ell \le s$ and $j_{\ell-1} < j \le j_\ell$.

Furthermore, we have that $k_\ell = n - j_\ell$, and so

$$0 = k_s < k_{s-1} < \ldots < k_1 < k_0 = n.$$

Then the multiplicity of the value γ_ℓ is

$$m_\ell := j_\ell - j_{\ell-1} = k_{\ell-1} - k_\ell.$$

We observe that $s_{j_\ell}(\alpha_1, \ldots, \alpha_n)$ is a sum of products of j_ℓ many roots of f. Since $v\alpha_{j_\ell} < v\alpha_{j_\ell+1}$ for $\ell < s$, the unique product of minimal value must be

$$\prod_{i=1}^{j_\ell} \alpha_i.$$

This shows that

$$va_{k_\ell} = vs_{j_\ell}(\alpha_1, \ldots, \alpha_n) = v \prod_{1 \le i \le j_\ell} \alpha_i = \sum_{1 \le i \le j_\ell} v\alpha_i. \tag{3.1}$$

Therefore,

$$va_{k_\ell} - va_{k_{\ell-1}} = \sum_{j=j_{\ell-1}+1}^{j_\ell} v\alpha_j = m_\ell \gamma_\ell,$$

Consequently,

$$\gamma_\ell = \frac{va_{k_\ell} - va_{k_{\ell-1}}}{j_\ell - j_{\ell-1}} = -\frac{va_{k_{\ell-1}} - va_{k_\ell}}{k_{\ell-1} - k_\ell}. \tag{3.2}$$

The above equation remains true if $va_{k_s} = \infty$, since $va_{k_{s-1}} < \infty$. In this case we have that $\gamma_s = \infty$.

By (3.2), the slope of the the generalized segment in $\mathbb{R} \times (\Gamma \cup \{\infty\})$ connecting the points (k_ℓ, va_{k_ℓ}) and $(k_{\ell-1}, va_{k_{\ell-1}})$ is equal to $-\gamma_\ell$. Thus we will be able to compute the values of all roots of f and their multiplicities once we are able to recognize the numbers k_ℓ from the values of the coefficients of f.

First we observe that for $0 < \ell \leq s$, the slope $-\gamma_\ell$ of the generalized segment from the point (k_ℓ, va_{k_ℓ}) to the point $(k_{\ell-1}, va_{k_{\ell-1}})$ is strictly smaller than the slope $-\gamma_{\ell-1}$ of the next generalized segment. By Corollary 3.1.3, the segments form the graph of an upward convex piecewise linear function on the real interval $[0, n]$ (if 0 is not a root of f) or on $[k_1, n]$ (if 0 is a root of f).

Now we determine the location of the remaining points (k, va_k). Assume that $k_\ell < k < k_{\ell-1}$, that is,

$$j_{\ell-1} = n - k_{\ell-1} < n - k < n - k_\ell = j_\ell.$$

Then the products of minimal value in $s_{n-k}(\alpha_1, \ldots, \alpha_n)$ are of the form $\prod_{i=1}^{j_{\ell-1}} \alpha_i$ times a product of $n - k - j_{\ell-1}$ many roots of value γ_ℓ. Hence, from (3.2) we obtain:

$$\begin{array}{rcl} va_k & \geq & va_{k_{\ell-1}} + (n - k - j_{\ell-1})\gamma_\ell \\ & = & va_{k_\ell} + (k_{\ell-1} - k_\ell)(-\gamma_\ell) + (k - k_{\ell-1})(-\gamma_\ell) \\ & = & va_{k_\ell} + (k - k_\ell)(-\gamma_\ell). \end{array}$$

This shows that the point (k, va_k) lies on or above the segment connecting (k_ℓ, va_{k_ℓ}) with $(k_{\ell-1}, va_{k_{\ell-1}})$ if $va_{k_{\ell-1}} \neq \infty$, and that $va_k = \infty$ otherwise.

We note that multiplying f with a nonzero leading coefficient a_n will only shift the graph up or down by va_n but will not change slopes nor roots. Therefore, we obtain the values of an arbitrary polynomial of degree n in exactly the same way as above.

We thus see that all points (i, va_i), $0 \leq i \leq n$, lie on or above the graph of the function we have constructed. Write

$$A := \{(i, va_i) \mid 1 \leq i \leq n \wedge a_i \neq 0\}.$$

By Proposition 3.1.4 and by part (c) of Proposition 3.1.1 we thus obtain:

The function described by the graph we have constructed is

$$\mathrm{NP}_A = \mathrm{NP}_{\phi(A)} : \mathbb{R} \to \Gamma \cup \{\infty\}.$$

Since the points in $\phi(A)$ are precisely the points such that the slope of NP_A changes at their respective first coordinates, we have that

$$\phi(A) = \{(k_\ell, va_{k_\ell}) \mid 1 \leq \ell \leq s \wedge a_{k_\ell} \neq 0\}.$$

We will denote this function by NP_f. We will refer to both $\mathcal{G}(\mathrm{NP}_f)$ (as a subset of $\mathbb{R} \times \Gamma$) and NP_f (as a function on $\mathbb{R}$) as the *Newton Polygon of f*.

The points $(k_\ell, va_{k_\ell}) = (k_\ell, \mathrm{NP}_f(k_\ell)) \in \mathbb{R} \times (\Gamma \cup \{\infty\})$, $0 \leq \ell \leq s$, are the vertices of NP_f, the generalized segments connecting the vertices are its faces. For $0 < \ell \leq s$, the respective positive integers $k_{\ell-1} - k_\ell$ are the length of the face. In terms of these notions, we have shown:

Theorem 3.2.2. *Take a polynomial $f \in K[x]$ as in (2.1). If the Newton Polygon of f has a face of length k with slope $-\gamma$, then f has exactly k many roots of value γ (counted with multiplicity). In other words,*

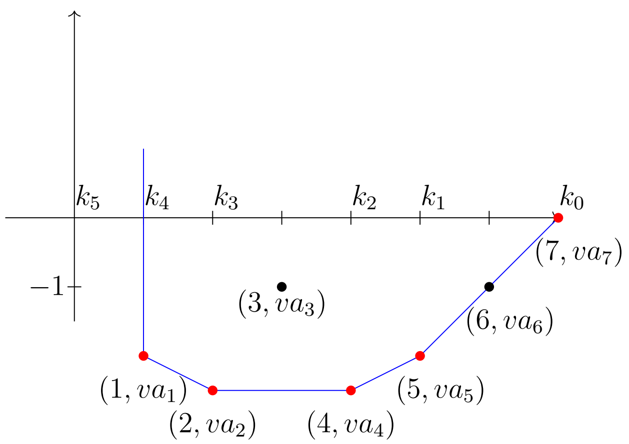
$$v(\alpha_i) = \mathrm{NP}_f(n - i) - \mathrm{NP}_f(n - i + 1), \tag{3.3}$$

with the convention $\infty - \infty := \infty$.

Example 3.2.3. Consider the field $\mathbb{Q}(\sqrt{2})$ with the 2-adic valuation v. Take the polynomial

$$f(x) = x^7 + \frac{x^6}{2} + \frac{x^5}{4} + \frac{x^4}{4\sqrt{2}} + \frac{x^3}{2} + \frac{x^2}{4\sqrt{2}} + \frac{x}{4}.$$

The Newton Polygon of f is represented by the blue graph in the following picture:



The vertices of NP_f are represented by the red dots on the graph and are of the form (k_i, va_{k_i}). Note that the corresponding function $\mathrm{NP}_f(k)$ has value ∞ for $0 \le k < 1$ and for $k > 7$, and that NP_f is a piecewise linear function on the real interval $[1, 7]$. From Equation (3.3) we know that the values of the roots of f are as follows:

$$v\alpha_7 = \mathrm{NP}_f(0) - \mathrm{NP}_f(1) = \infty - (-2) = \infty = \gamma_5,$$

$$v\alpha_6 = \mathrm{NP}_f(1) - \mathrm{NP}_f(2) = -2 - (-2\tfrac{1}{2}) = \tfrac{1}{2} = \gamma_4,$$

$$v\alpha_5 = \mathrm{NP}_f(3) - \mathrm{NP}_f(2) = 0 = \gamma_3,$$

$$v\alpha_4 = \mathrm{NP}_f(4) - \mathrm{NP}_f(3) = 0 = \gamma_3,$$

$$v\alpha_3 = \mathrm{NP}_f(5) - \mathrm{NP}_f(4) = -\tfrac{1}{2} = \gamma_2,$$

$$v\alpha_2 = \mathrm{NP}_f(6) - \mathrm{NP}_f(5) = -1 = \gamma_1,$$

$$v\alpha_1 = \mathrm{NP}_f(7) - \mathrm{NP}_f(6) = -1 = \gamma_1.$$

A simple application of Theorem 3.2.2 is given in the following lemma. It can also be proved directly through the properties of the symmetric functions (see [9, Lemma 5.6] for details). However, using the Newton Polygon allows us to give simple, elementary proofs, and to give more detailed assertions.

Lemma 3.2.4. *Let $f \in K[x]$ be a polynomial as in (2.1) such that $v\alpha_i \le v\alpha_{i+1}$ for $1 \le i < n$.*

(a) *If f is monic, then $f \in \mathcal{O}_K[x]$ if and only if all roots of f are integral.*

(b) *Suppose that among all coefficients of minimal value, a_{l_0} is the one with smallest index, and a_{l_1} is the one with largest index. Then f has l_0 many roots of positive value, $l_1 - l_0$ roots of value 0 and $n - l_1$ roots of negative value.*

(c) *Let l_0 and l_1 be as in (b) and take $l_0 \le l \le l_1$. Then*

$$va_l - va_n = \sum_{i=1}^{n-l} v\alpha_i,$$

and

$$va_0 - va_l = \sum_{i=n-l+1}^{n} v\alpha_i.$$

(*d*) *For every* $l \in \{1, \dots, n\}$ *we have that*

$$-\max_{1\le i\le n} \frac{va_n - va_{n-i}}{i} \le v\alpha_l \le -\min_{1\le i\le n} \frac{va_i - va_0}{i},$$

and the minimum and maximum are achieved for some roots of f*.*

(*e*) *If* α_l *is such that* $v\alpha_l \neq v\alpha_i$ *for* $i \neq l$*, then* $v\alpha_l \in vK$*. More generally, if there are precisely* t *roots of* f *with value* $v\alpha_l$ *and if* vK *is* t*-divisible, then* $v\alpha_l \in vK$*.*

Proof. To prove part (*a*), take a monic polynomial f of degree n. Then $\mathrm{NP}_f(n) = 0$.

If f has only integral roots, then all the slopes of NP_f are non-positive, hence $\mathrm{NP}_f(i) \ge 0$ for $0 \le i \le n$. Since the value of each coefficient of f lies on or above NP_f, we obtain that f has integral coefficients.

Conversely, assume that f has at least one root of negative value. Then in particular, the rightmost slope of NP_f is positive. Since $\mathrm{NP}_f(n) = 0$, we must have that $\mathrm{NP}_f(i) < 0$ for some $i \in \{0, \dots, n-1\}$, which implies that at least one of the coefficients of f is not integral. This finishes the proof of part (*a*).

Part (*b*) follows from the fact that all the slopes on the left side of the coordinate l_0 are negative, along the interval $[l_0, l_1]$ the slope is equal to zero, and on the right side of l_1 the slopes are positive.

We will only prove the second inequality for $l = l_0$ of part (*c*), since the remaining assertions are proved analogously. Note that by the previous paragraph, a_l is a vertex of NP_A. We have that

$$va_0 - va_l = \mathrm{NP}_f(0) - \mathrm{NP}_f(l) = \sum_{i=1}^{l} (\mathrm{NP}_f(i-1) - \mathrm{NP}_f(i)) = \sum_{i=n-l+1}^{n} \alpha_i.$$

We know that a root of maximal (resp. minimal) value corresponds to the leftmost (resp. rightmost) slope of NP_f. By construction, the leftmost slope of NP_f is equal to

$$\frac{va_{k_{s-1}} - va_0}{k_{s-1}} = \min_{1\le i\le n} \frac{va_i - va_0}{i}.$$

This means that the largest value that a root of f admits is precisely

$$-\min_{1\le i\le n} \frac{va_i - va_0}{i}.$$

We claim that the rightmost slope of NP_f is of the form

$$\frac{va_n - va_{k_1}}{n - k_1} = \max_{1 \le i \le n} \frac{va_n - va_{n-i}}{i}.$$

Indeed, if there were a larger slope than the one on the left hand side, then a_{k_1} would not be a vertex of NP_f, contradiction. Since this is the largest slope of NP_f, it corresponds to the root of f whose value is minimal. This value is equal to

$$-\max_{1 \le i \le n} \frac{va_n - va_{n-i}}{i}.$$

This proves part (d).

The assumption of part (e) means that the length of the slope which corresponds to $v\alpha_l$ has length t, and thus there exists $i \in \{0, \dots, s\}$ such that

$$v\alpha_l = \frac{va_{k_{i+1}} - va_{k_i}}{t} \in vK.$$

□

Chapter 4

Applications of the Newton Polygon

Let (K, v) be an arbitrary valued field and $f \in K[x]$. In the previous chapter, we showed in Theorem 3.2.2 that if NP_f has a face with slope γ and multiplicity t, then f has precisely t roots (counted with multiplicities) of value $-\gamma$. In this chapter, we will study connections between Newton Polygons of polynomials which are sufficiently close to each other. We will then employ Theorem 3.2.2 to formulate results on continuity of roots and on values of roots.

In this chapter, when we speak of "the number of roots" of a polynomial, we mean the number of roots counted with their multiplicities.

4.1 Values of roots

As before, we assume that the integers k_ℓ and the slopes γ_ℓ for the polynomial $f \in K[x]$ are defined as in Notation 3.2.1. The following theorem gives us the main tool for studying connections between values of roots of polynomials which are close to each other.

Theorem 4.1.1. *Consider the polynomials f and g as in (2.1) with f monic and $m \geq n$. Fix some $\varepsilon \geq 0$, assume that $v(f-g) > n\varepsilon$ and that the set*

$$\{\ell \in \{1 \ldots, s\} \mid \gamma_\ell \leq \varepsilon\}$$

is nonempty. If ℓ_ε is the maximum of this set, then $\mathrm{NP}_f(k) = \mathrm{NP}_g(k)$ for $k \in \{k_{\ell_\varepsilon}, \ldots, n\}$.

Proof. Fix any index $\ell \in \{0, \dots, \ell_\varepsilon\}$. By (3.1) we have that

$$va_{k_\ell} = \sum_{1 \le i \le j_\ell} v\alpha_i \le j_{\ell_\varepsilon}\gamma_{\ell_\varepsilon} \le n \cdot \max\{0, \gamma_{\ell_\varepsilon}\} \le n\varepsilon.$$

Since $v(a_{k_\ell} - b_{k_\ell}) \ge v(f - g) > n\varepsilon$, it follows that $va_{k_\ell} = vb_{k_\ell}$. As NP_f is upward convex, for $k_\ell \le k \le n$ we have that $\mathrm{NP}_f(k) \le \max\{va_{k_\ell}, va_n\} \le n\varepsilon$. Since the point (k, va_k) lies on or above the polygon, we have that $va_k \ge \mathrm{NP}_f(k)$. Hence, $vb_k \ge \min\{va_k, v(a_k - b_k)\} \ge \min\{\mathrm{NP}_f(k), n\varepsilon\} = \mathrm{NP}_f(k)$, so that also the point (k, vb_k) lies on or above NP_f. This shows that the points $(k_\ell, vb_{k\ell})$ are vertices of NP_g. Therefore, from $k = k_{\ell_\varepsilon}$ to $k = n$ we have that $\mathrm{NP}_f(k) = \mathrm{NP}_g(k)$.

□

The above theorem tells us that along a certain interval, the Newton Polygons of the polynomials f and g have the same vertices and slopes. This yields a connection between the values of the roots of f and g. To give more details about this connection, we will require a number of lemmas.

Lemma 4.1.2. *Take f, g as in (2.1) with f monic. If $v(f - g) > 0$, then $\mathrm{NP}_g(n) = \mathrm{NP}_f(n) = 0$.*

Proof. If $\gamma_1 \le 0$, then f and g satisfy the assumptions of Theorem 4.1.1 with $\varepsilon = 0$. Hence, $\mathrm{NP}_g(n) = \mathrm{NP}_f(n) = 0$. Assume that $\gamma_1 > 0$. Since f is monic, by part (a) of Lemma 3.2.4 we have that $va_k \ge 0$ for all $k \in \{0, \dots, n\}$. Note that $v(b_k - a_k) \ge v(f - g) > 0$, so we also have that $vb_k \ge 0$ for all $k \in \{0, \dots, m\}$ and

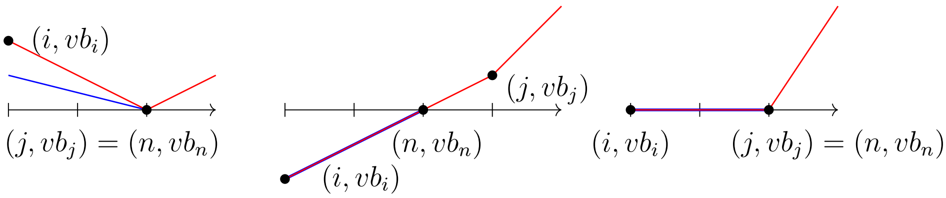
$$0 = \mathrm{NP}_f(n) = va_n = vb_n \ge \mathrm{NP}_g(n).$$

Let (i, vb_i) and (j, vb_j) be the vertices located on the left and on the right end of the face of NP_g that contains the point $(n, \mathrm{NP}_g(n))$. Then

$$\mathrm{NP}_g(n) \ge \min\{vb_i, vb_j\} \ge 0.$$

Hence, $\mathrm{NP}_g(n) = 0$. □

The following picture illustrates the situation in Lemma 4.1.2. We see the possible locations of the vertices on the ends of the face of NP_g that contains the point $(n, \mathrm{NP}_g(n))$, depending on the value γ_1. Note that for $\gamma_1 \ge 0$ we must have that the point $(j, vb_j) = (n, vb_n)$ is always a vertex of NP_g. This situation is studied in more detail in Lemma 4.1.4.

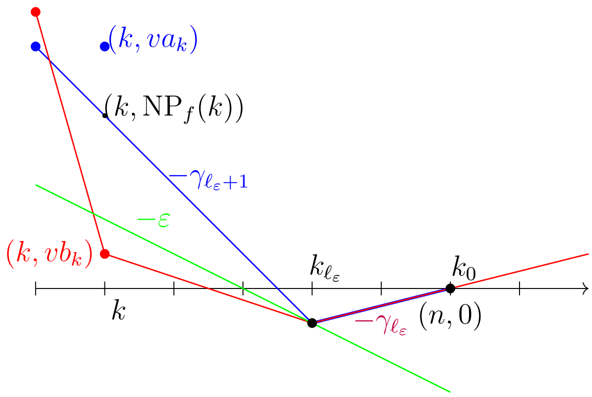


Lemma 4.1.3. *Take f and g satisfying the assumptions of Theorem 4.1.1. Then all the slopes of NP_g located on the left of the coordinate k_{ℓ_ε} are strictly smaller than $-\varepsilon$. In particular, $(k_{\ell_\varepsilon}, vb_{k_{\ell_\varepsilon}})$ is a vertex of NP_g.*

Proof. By Theorem 4.1.1, $\mathrm{NP}_f(k) = \mathrm{NP}_g(k)$ for $k \in \{k_{\ell_\varepsilon}, \dots, n\}$. We claim that proving the first assertion will prove the second assertion. Indeed, the slope located on the right of k_{ℓ_ε} is equal to $-\gamma_{\ell_\varepsilon}$, whereas by the first assertion, the slope located on the left of this coordinate is strictly smaller than $-\varepsilon \leq -\gamma_{\ell_\varepsilon}$. This means that the point $(k_{\ell_\varepsilon}, \mathrm{NP}_g(k_{\ell_\varepsilon}))$ is a vertex of NP_g, which then by definition must be equal to the point $(k_{\ell_\varepsilon}, vb_{k_{\ell_\varepsilon}})$.

Suppose that the face of NP_g which is located on the left of the coordinate k_{ℓ_ε} has slope greater than or equal to $-\varepsilon$. Let (k, vb_k) be the left vertex of this face for some $0 \leq k < k_{\ell_\varepsilon}$.

Figure 4.1: The Newton Polygon of f (blue segments) and the supposed Newton Polygon of g (red segments). The line going through the point $(k_{\ell_\varepsilon}, va_{k_{\ell_\varepsilon}})$ with slope $-\varepsilon$ is drawn in green.



On the one hand, we have that

$$vb_k \leq \sum_{i=1}^{\ell_\varepsilon} (k_{i-1} - k_i)\gamma_i + (k_{\ell_\varepsilon} - k)\varepsilon \leq \sum_{i=1}^{\ell_\varepsilon} (k_{i-1} - k_i)\varepsilon + (k_{\ell_\varepsilon} - k)\varepsilon = (n-k)\varepsilon \leq n\varepsilon.$$

On the other hand, the supposed slope of NP_g located to the left of the coordinate k_{ℓ_ε} is greater than or equal to $-\varepsilon$, whereas the corresponding slope of NP_f is strictly smaller than $-\varepsilon$. Since $\mathrm{NP}_g(k_{\ell_\varepsilon}) = \mathrm{NP}_f(k_{\ell_\varepsilon})$, this means that

$$vb_k = \mathrm{NP}_g(k) < \mathrm{NP}_f(k) \leq va_k.$$

As a result,

$$vb_k = v(a_k - b_k) \geq v(f - g) > n\varepsilon,$$

which gives us a contradiction.

□

Lemma 4.1.4. *Take polynomials f and g as in (2.1), with f monic and $m \geq n$. Take any $\varepsilon \geq 0$ and assume that $v(f-g) > n\varepsilon$. If $\varepsilon \geq (1 - \frac{m}{n})\gamma_1$, then all slopes of NP_g along the interval $[n, m]$ are strictly greater than $-\gamma_1$. In particular, if under these assumptions NP_f and NP_g coincide along some interval $[k, n]$ for $k < n$, then (n, vb_n) is a vertex of NP_g.*

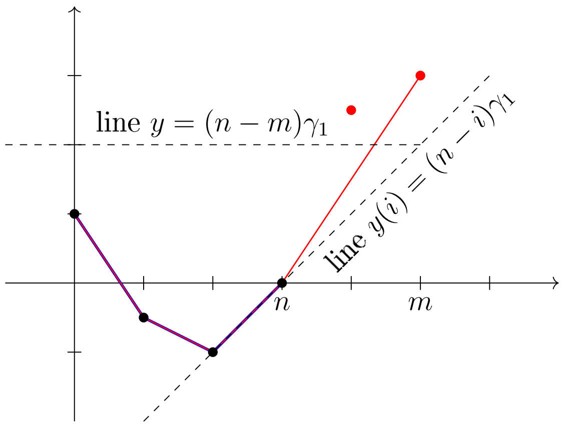
Proof. Assume without loss of generality that $m > n$ since otherwise the assertion holds trivially.

By Lemma 4.1.2, $\mathrm{NP}_g(n) = 0$. Since $vb_i > 0$ for $n < i \leq m$, NP_g has positive slope along the interval $[n, m]$. Thus the assertion is proved if $\gamma_1 \geq 0$, since in this case the corresponding slope $-\gamma_1$ located on the left of the coordinate n would be non-positive.

Assume now that $\gamma_1 < 0$ and that $\varepsilon \geq (1 - \frac{m}{n})\gamma_1$. The assertions of the lemma are satisfied if and only if for all $i \in \{n+1, \ldots, m\}$ the point (i, vb_i) lies above the line going through the point $(n, \mathrm{NP}_g(n))$ with slope $-\gamma_1$ (note that by the upward convexity of NP_g those points cannot lie below that line). This in turn is equivalent to saying that for all $i > n$ the segment connecting $(n, \mathrm{NP}_g(n))$ and (i, vb_i) has slope strictly greater than $-\gamma_1$, that is,

$$vb_i > \mathrm{NP}_g(n) - (i - n)\gamma_1. \tag{4.1}$$

Figure 4.2: The Newton Polygons of f (blue segments) and g_j (red segments) and the corresponding vertices. It visualizes how the condition $vb_i > (n-i)\gamma_1$ is indeed equivalent to stating that g has precisely as many roots of value γ_1 as f. Compared to the condition $vb_i > (n-i)\gamma_1$, the condition $vb_i > (n-m)\gamma_1$ presents a bound for vb_i that is not dependent on i, but possibly much larger.



By assumption and since $\mathrm{NP}_g(n) \le vb_n = 0$, we have that

$$vb_i \ge v(f-g) > n\varepsilon \ge (n-m)\gamma_1 \ge (n-i)\gamma_1 \ge \mathrm{NP}_g(n) - (i-n)\gamma_1.$$

Therefore (4.1) holds, and so our lemma is proved. □

Consider the ultrametric space induced by the valuation v on K. Take $a \in K$ and $\gamma \in vK$. Recall from Definition 1.2.2 that the open ultrametric ball of radius γ centered at a is the following set:

$$B^{\circ}_{\gamma}(a) := \{b \in K \mid v(a-b) > \gamma\}.$$

Similarly, we define the set

$$S_{\gamma}(a) := \{b \in K \mid v(a-b) = \gamma\}$$

and call it *the ultrametric sphere of radius γ centered at a*.

We define $n_b(f,\gamma,a)$ and $n_s(f,\gamma,a)$ to be the number of roots of f (counted with multiplicities) in $B^{\circ}_{\gamma}(a)$ and $S_{\gamma}(a)$, respectively. We will often consider those numbers in the case where $a = 0$. For brevity, we will write $n_b(f,\gamma) := n_b(f,\gamma,0)$ and $n_s(f,\gamma) := n_s(f,\gamma,0)$. Then $n_s(f,\gamma)$ is the number of roots of f with value γ, and $n_b(f,\gamma)$ is the number of roots of f with value strictly greater than γ.

Theorem 4.1.5. *Let the polynomials f and g be as in Theorem 4.1.1. Then:*

(a) $n_s(g,\gamma) = n_s(f,\gamma)$ *if* $\gamma_1 < \gamma < \varepsilon$ *or if* $\gamma = \varepsilon$.

(b) $n_s(g,\gamma_1) \geq n_s(f,\gamma_1)$ *and equality holds if and only if the point* (n, vb_n) *is a vertex of* NP_g.

(c) $n_b(g,\gamma) = n_b(f,\gamma)$ *for* $\gamma_1 \leq \gamma \leq \varepsilon$ *and* $n_b(g,\gamma) \geq n_b(f,\gamma)$ *for* $\gamma < \gamma_1$.

(d) $k_{\ell_\varepsilon} = n_b(f,\gamma_{\ell_\varepsilon}) = n_b(g,\gamma_{\ell_\varepsilon}) = n_b(g,\varepsilon) = n_b(f,\varepsilon)$. *In particular, if* $\gamma'_{\ell'_\varepsilon}$ *is chosen for* g *in the same manner as* $\gamma_{\ell_\varepsilon}$ *was chosen for* f*, then* $\gamma'_{\ell'_\varepsilon} = \gamma_{\ell_\varepsilon}$.

(e) g *has* $m - k_1$ *many roots of value* $\leq \gamma_1$.

(f) g *has* $m - n$ *many roots of value* $< \gamma_1$ *if and only if the point* (n, vb_n) *is a vertex of* NP_g.

Proof. By Theorem 4.1.1 we have that $\mathrm{NP}_f(k) = \mathrm{NP}_g(k)$ for $k \in \{k_{\ell_\varepsilon}, \ldots, n\}$. This fact combined with Lemma 4.1.3 yields the following observations:

(i) Along the segment $[k_{\ell_\varepsilon}, k_1]$, all the vertices and slopes of NP_g are precisely the same as the respective vertices and slopes of NP_f.

(ii) All the slopes of both NP_g and NP_f along the interval $[0, k_{\ell_\varepsilon}]$ are strictly smaller than $-\varepsilon \leq -\gamma_{\ell_\varepsilon}$. The slopes of NP_g and NP_f located on the right of the coordinate k_{ℓ_ε} are greater than or equal to $-\gamma_{\ell_\varepsilon}$.

(iii) NP_g has a face of slope $-\gamma_1$ along the interval $[k_1, k']$ for some $n \leq k' \leq m$. This face contains the corresponding face of NP_f of slope $-\gamma_1$ which runs along the interval $[k_1, n]$. The lengths of those two faces are equal if and only if $k' = n$. This happens if and only if the point (n, vb_n) is a vertex of NP_g.

(iv) With k' as in (iii), all the slopes of NP_g along the interval $[k', m]$ are strictly greater than $-\gamma_1$.

We will now combine those facts with Theorem 3.2.2.

Assertion (a) for $\gamma_1 < \gamma \leq \varepsilon$ follows from observations (i) and (ii). The remaining case is $\gamma_1 = \gamma = \varepsilon \geq 0$ because by assumption, $\gamma_1 \leq \varepsilon$. Then in particular $\varepsilon \geq (1 - \frac{m}{n})\gamma_1$. We can thus use Lemma 4.1.4 to find that $k' = n$ for k' as in observation (iii). This finishes the proof of assertion (a).

We will now prove assertion (c). Note that $n_b(f,\varepsilon) = n_b(g,\varepsilon)$ by observation (ii). For $\gamma_1 \leq \gamma < \varepsilon$ we have that $n_b(f,\gamma) = n_b(f,\varepsilon) + \sum_{i\in I} n_s(f,\gamma_i)$, where $I = \{2 \leq i \leq k_{\ell_\varepsilon} \mid \gamma < \gamma_i\}$ is a (possibly empty) set of indices. By assertion (a), $n_s(f,\gamma_i) = n_s(g,\gamma_i)$ for all $i \in I$ and $n_s(g,\delta) = 0$ for each value δ such that $\gamma_1 < \delta \leq \varepsilon$ and δ is not of the form γ_i for some $i \in I$. Thus,

$$n_b(g,\gamma) = n_b(g,\varepsilon) + \sum_{i\in I} n_s(g,\gamma_i) = n_b(f,\varepsilon) + \sum_{i\in I} n_s(f,\gamma_i) = n_b(f,\gamma).$$

For $\gamma < \gamma_1$, we have that $n_b(f,\gamma) = n \leq n_b(g,\gamma)$.

Assertion (d) follows from (ii), assertion (b) follows from (iii), and assertions (e) and (f) follow from (iii) and (iv). □

The above computations only need that for all i either $va_i = vb_i$ or $va_i, vb_i \geq n\varepsilon$. Thus the results can be generalized, as in [6], to the case where f and g are polynomials over two different valued fields with their respective value groups contained in a common ordered Abelian group.

Corollary 4.1.6. *Let the polynomials f and g satisfy the assumptions of Theorem 4.1.1. Then $n_s(g,\gamma) = n_s(f,\gamma)$ for all γ such that $\gamma_1 \leq \gamma \leq \varepsilon$ if and only if (n, vb_n) is a vertex of NP_g. This holds in particular if $m = n$ or, more generally, if $\varepsilon \geq (1-\frac{m}{n})\gamma_1$. Moreover, if $m = n$, then $n_s(g,\gamma) = n_s(f,\gamma)$ for all $\gamma \leq \varepsilon$.*

Proof. The first assertion follows directly from parts (a) and (b) of Theorem 4.1.5. The particular case for $\varepsilon \geq (1-\frac{m}{n})\gamma_1$ holds by Lemma 4.1.4. Note that for $m = n$ this condition reads $\varepsilon \geq 0$, which was our original assumption on ε. If $m = n$, then by part (f) of Theorem 4.1.5 we have that $n_s(g,\gamma) = n_s(f,\gamma) = 0$ for $\gamma < \gamma_1$. □

Note that by Theorem 4.1.5 there is a value-preserving bijection between the respective multisets of roots of f and g whose values do not exceed ε (in a multiset of roots of f, we take t copies of α if a root α of f has multiplicity t). However, when we speak of sets in the usual sense, the corresponding value-preserving pairing may not even be a mapping. For example, a root of multiplicity 2 may be sent to two distinct roots of multiplicity 1.

Observe further that for $c \in K$ the map $x \mapsto x - c$ induces a bijection between the roots of f and those of f_c (as in Definition 1.6.2), and a bijection between $B^\circ_\gamma(c)$ and $B^\circ_\gamma(0)$. Thus $n_b(f,\gamma,c) = n_b(f_c,\gamma,0)$.

Corollary 4.1.7. *Take $\varepsilon > 0$ and $c \in K$. Let $f, g \in K[x]$ be two monic polynomials of degree n. If*

$$v(f-g) > n\varepsilon - \min\{0,(n-1)vc\}, \tag{4.2}$$

then $n_b(f,\varepsilon,c) = n_b(g,\varepsilon,c)$.

Proof. From the observation before the corollary we have that

$$n_b(f,\varepsilon,c) = n_b(f_c,\varepsilon,0) \quad \text{and} \quad n_b(g,\varepsilon,c) = n_b(g_c,\varepsilon,0).$$

By Lemma 1.6.3 and by (4.2) we have that

$$v(f_c - g_c) \geq v(f-g) + \min\{0,(n-1)vc\} > n\varepsilon.$$

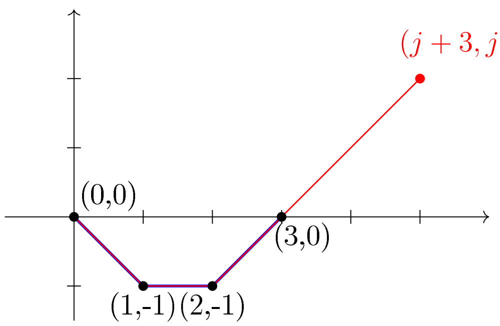
By part (c) of Theorem 4.1.5, $n_b(f_c,\varepsilon,0) = n_b(g_c,\varepsilon,0)$, thus $n_b(f,\varepsilon,c) = n_b(g,\varepsilon,c)$. □

Remark 4.1.8. Observe that the assumption $\varepsilon \geq 0$ already implies that $\varepsilon \geq (1-\frac{m}{n})\gamma_1$ if $\gamma_1 \geq 0$. Hence, this assumption is only relevant if $\gamma_1 < 0$.

If we wish to have a bound that does not require computing the value of a root of f, then we may use the fact that $\gamma_1 \geq vf$ (cf. Lemma 2.1.5), thus $(1-\frac{m}{n})vf \geq (1-\frac{m}{n})\gamma_1$. In this case, the condition in the above lemma can be replaced by $\varepsilon \geq (1-\frac{m}{n})vf$.

Note that in the case where $\gamma_1 < 0$, the bound for ε depends also on the degree of the polynomial g. We will now show that it is impossible to specify a bound which is independent of m. Consider the field $\mathbb{Q}$ with the 2-adic valuation v and the polynomial $f(x) = 1+\frac{1}{2}x+\frac{1}{2}x^2+x^3$. For $j \geq 1$ we define $g_j(x) = f(x) + 2^j x^{j+3}$. Then the rightmost face of the Newton Polygons of f and g_j has slope 1. The length of that face is 1 for f and $j+1$ for g_j. This means that f has one root of value $\gamma_1 = -1$, whereas each g_j has $j+1$ roots of value -1. However, $v(f-g_j) = j$, which means that for every $\varepsilon \in v\mathbb{Q} = \mathbb{Z}$ there exists j such that $v(f-g_j) > n\varepsilon$, but f and g_j do not have the same number of roots of value γ_1.

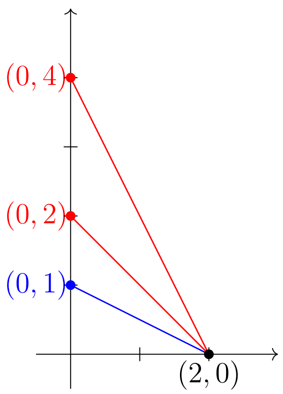
Figure 4.3: The Newton Polygons of f (blue segments) and g_j (red segments) and the corresponding vertices.



By Theorem 4.1.5 we have that some roots of polynomials which are close to each other will have equal values under a suitable pairing. Consider the following situation: we fix the polynomial f of degree n and the corresponding values γ_i as before. We then simultaneously consider all polynomials g of degree $m \geq n$ such that $v(f-g) > 0$. We know that as soon as the value $v(f-g)$ passes the value $n\gamma_i$, we have that $n_s(g, \gamma_i) \geq n_s(f, \gamma_i)$. However, Theorem 4.1.5 gives us no information on $n_s(g, \gamma_i)$ if $0 < v(f-g) < n\gamma_i$. One may hope that there is a result on the "continuity of values of roots": if the value $v(f-g)$ approaches $n\gamma_i$ from below, then some roots of f and g will have values which are getting closer and closer to each other under a suitable pairing. However, this is not the case. The values of roots can be arbitrarily far from each other as long as $v(f-g) \leq n\gamma_i$, only to suddenly become equal once the value $v(f-g)$ jumps above $n\gamma_i$. In fact, we can find polynomials g_j such that $v(f-g_j) = n\gamma_i$ and for every choice of roots α of f and β_j of g_j, the set $\{|v\alpha - v\beta_j| \mid j \geq 1\}$ will be cofinal in $v\tilde{K}$. This is illustrated in the following example.

Example 4.1.9. Consider the field $\mathbb{Q}$ with the 2-adic valuation v and extend v to $\mathbb{Q}(\sqrt{2})$. Take $f(x) = x^2 - 2$, then both roots of f have value $\gamma_1 = \frac{1}{2}$. Consider the polynomials $g_j(x) = x^2 - 2^{2j}$, where j is a positive integer. We have that $v(f - g_j) = 1 = 2\gamma_1$, but both roots of g_j have value j.

Figure 4.4: The Newton Polygons of f (blue) and g_j (red) for $j = 1, 2$.



4.2 Continuity of roots

In this section, we will present alterations of results found in [2], as well as results from [6] and [7].

Krasner's constant of an element $\alpha \in K^{\text{sep}} \setminus K$ is defined as follows:

$$\text{kras}_K(\alpha) = \max\{v(\alpha - \sigma\alpha) \mid \sigma \in \text{Gal}\, K \wedge \sigma\alpha \neq \alpha\}.$$

Note that if α is a root of a separable polynomial $f \in K[x]$, then $\mathrm{kras}(f) \geq \mathrm{kras}_K(\alpha)$.

The first theorem stated in this section is an application of the Newton Polygon. Its formulation and proof are alterations of [2, Theorem 1]. At the cost of modifying the bound given in [2], we are able to drop the assumptions on the polynomials in question to be of the same degree, both monic and separable, and to have integral coefficients. For the original formulation of this theorem, see Section 4.3.

To prove the second part of the theorem, we will employ the following version of Krasner's Lemma (see e.g. [4, 16.8], [5, Theorem 4.1.7]).

Lemma 4.2.1. *Let (K, v) be a Henselian valued field. Then for every element $\alpha \in K^{\mathrm{sep}}$ the following holds: if $\beta \in K^{\mathrm{sep}} \setminus K$ satisfies $v(\alpha - \beta) > \mathrm{kras}_K(\alpha)$, then $\alpha \in K(\beta)$.*

For a root α_k of f we denote by t_k its multiplicity.

Theorem 4.2.2. *Let (K, v) be a valued field and take $f, g \in K[x]$, written as in (2.1) with f monic and $m \geq n$. Assume that for $\varepsilon \geq \max\{0, \mathrm{kras}(f)\}$ we have that*

$$v(f - g) > n\varepsilon - \deg(f - g)\gamma^*(f). \tag{4.3}$$

Then, after suitably rearranging indices, for every $k \in \{1, \ldots, n\}$ we have that $v(\alpha_k - \beta_k) > t_k\varepsilon$.

If in addition (K, v) is Henselian and f and g are separable, then for each k we have that $K(\alpha_k) \subseteq K(\beta_k)$.

Proof. Choose a root α of f and consider $f_\alpha(x) := f(x+\alpha)$, $g_\alpha(x) := g(x+\alpha)$. Denote by a'_i and b'_i the respective coefficients of f_α and g_α. We will now prove a number of results on the Newton Polygons of f_α and g_α.

If α is a root of f of multiplicity $t \geq 1$, then 0 is a root of f_α of multiplicity t. Hence for $0 \leq i \leq t-1$ we have that $\mathrm{NP}_{f_\alpha}(i) = va'_i = \infty$. Moreover,

$$\begin{cases} \mathrm{NP}_{f_\alpha}(t) & = & va'_t = v(s_{n-t}(\alpha_1 - \alpha, \ldots, \alpha_n - \alpha)) \\ & = & v\left(\sum_{i_1 < i_2 < \ldots < i_{n-t}} (\alpha_{i_1} - \alpha) \cdot \ldots \cdot (\alpha_{i_{n-t}} - \alpha)\right) \\ & = & v\left(\prod_{j \in J} (\alpha_j - \alpha)\right) = \sum_{j \in J} v(\alpha - \alpha_j) \leq (n-t)\varepsilon, \end{cases} \tag{4.4}$$

where $J \subset \{1, \ldots, n\}$ is the set of all $n - t$ indices j such that $\alpha_j \neq \alpha$. By Lemma 1.6.3 we have that

$$\begin{aligned} v(f_\alpha - g_\alpha) & \geq v(f - g) + \min\{0, \deg(f - g)v\alpha\} \\ & \geq v(f - g) + \deg(f - g)\gamma^*(f) > n\varepsilon. \end{aligned}$$

Therefore, for $0 \leq i \leq t-1$ we have that

$$\mathrm{NP}_{g_\alpha}(i) = vb'_i = v(b'_i - a'_i) \geq v(f_\alpha - g_\alpha) > n\varepsilon. \tag{4.5}$$

Assume now that f is not purely inseparable. We apply Theorem 4.1.1 to the polynomials f_α and g_α and the value ε. Using the notation of that theorem, we see that $\gamma_s = \infty > \varepsilon$. Since f has at least two distinct roots, the property $\varepsilon \geq \mathrm{kras}(f)$ implies that $\varepsilon \geq \gamma_{s-1}$. Hence $\ell_\varepsilon = s-1$, and since 0 is a root of f_α of multiplicity t, we have that $k_{\ell_\varepsilon} = t$. As a result,

$$\mathrm{NP}_{f_\alpha}(i) = \mathrm{NP}_{g_\alpha}(i) \quad \text{for} \quad t \leq i \leq n. \tag{4.6}$$

For the time being, we write the indices of roots of g in such a way that

$$v(\beta_1 - \alpha) \geq v(\beta_2 - \alpha) \geq \ldots \geq v(\beta_m - \alpha). \tag{4.7}$$

Then $\beta_1 - \alpha, \ldots, \beta_t - \alpha$ are the t roots of g_α whose value exceeds that of the remaining roots of g_α. This means that their values correspond to the leftmost slopes of NP_{g_α}. In particular, we have that

$$v(\beta_t - \alpha) = \mathrm{NP}_{g_\alpha}(t-1) - \mathrm{NP}_{g_\alpha}(t). \tag{4.8}$$

By Equation (4.6) with $i = t$ and Equation (4.4) we have that

$$\mathrm{NP}_{g_\alpha}(t) = \mathrm{NP}_{f_\alpha}(t) \leq (n-t)\varepsilon. \tag{4.9}$$

Taking Equation (4.5) with $i = t-1$, we obtain that $\mathrm{NP}_{g_\alpha}(t-1) > n\varepsilon$. We combine this fact with Equations (4.8) and (4.9). As a result,

$$v(\beta_t - \alpha) = \mathrm{NP}_{g_\alpha}(t-1) - \mathrm{NP}_{g_\alpha}(t) > n\varepsilon - (n-t)\varepsilon = t\varepsilon.$$

By our choice of indices, we must have that $v(\beta_i - \alpha) > t\varepsilon$ for $1 \leq i \leq t$. Consider the set $J = \{j \in \{1, \ldots, n\} \mid \alpha_j = \alpha\}$ containing t many elements. We renumber the roots $\beta_1, \ldots, \beta_t$ by the indices in J so that they are paired with the roots α_j, $j \in J$. Then we have that $v(\beta_j - \alpha_j) > \varepsilon$ for all $j \in J$.

To the roots α_j, $j \in J$ we have now assigned the corresponding roots β_j. We claim that if we repeat this construction for another root $\alpha := \alpha_l$, $l \notin J$, then none of the so assigned roots β_l can be equal to β_j for any $j \in J$. Suppose that $\beta_l = \beta_j$ for some $j \in J$. Since $\varepsilon \geq \mathrm{kras}(f)$, we have that

$$v(\alpha_j - \alpha) \geq \min\{v(\alpha_j - \beta_j), v(\alpha - \beta_j)\} > \varepsilon \geq v(\alpha_j - \alpha).$$

We have now shown that for every root α of f of multiplicity t there exist at least t roots of g which satisfy our claim. Moreover, the argument above yields that those roots of g cannot be assigned to a root distinct from α. We can thus renumber the roots of g such that $v(\alpha_k - \beta_k) > \varepsilon$, assigning indices from $\{n+1, \ldots, m\}$ to the roots of g which were not chosen to be paired with any root of f.

If f is purely inseparable, then we have that $\mathrm{NP}_{f_\alpha}(i) = \infty$ for $i < n$, $\mathrm{NP}_{f_\alpha}(n) = 0$. Recall from (4.5) with $i = t-1 = n-1$ that $\mathrm{NP}_{g_\alpha}(n-1) > n\varepsilon$. Moreover, we have that $v(b'_n - a'_n) \geq v(f_\alpha - g_\alpha) > 0$, hence $\mathrm{NP}_{g_\alpha}(n) = vb'_n = 0$. We write the indices $1, \ldots, n$ of roots of g as in (4.7). Then

$$v(\beta_n - \alpha_k) = \mathrm{NP}_{g_\alpha}(n-1) - \mathrm{NP}_{g_\alpha}(n) > n\varepsilon.$$

Hence, also in the case where f is purely inseparable, we can renumber the roots of g in such a way that $v(\alpha_k - \beta_k) > \varepsilon \geq \mathrm{kras}(f)$.

To prove the last assertion, observe that the separability of f together with the above property implies that $v(\alpha_k - \beta_k) > \varepsilon \geq \mathrm{kras}_K(\alpha_k)$. Hence by Lemma 4.2.1, $K(\alpha_k) \subseteq K(\beta_k)$ for each k. This finishes the proof. □

Remark 4.2.3. We claim that the above theorem remains true if we replace "$>$" by "$\geq$" in condition (4.3) and the subsequent assertion, while also replacing "$\geq$" by "$>$" in the bound for ε. We will have a closer look at the proof of Theorem 4.2.2 and step by step consider the changes that the above replacements yield.

In this case, the last inequality in (4.4) becomes strict, thus stating that

$$\mathrm{NP}_{f_\alpha}(t) < (n-t)\varepsilon$$

as long as $n > t$, that is, f is not purely inseparable. In the purely inseparable case, both sides are equal to zero. The weak inequality that has now appeared in condition (4.3) means that (4.5) now reads

$$v(f_\alpha - g_\alpha) \geq n\varepsilon.$$

In particular, $\mathrm{NP}_{g_\alpha}(t-1) \geq n\varepsilon$. The property $\varepsilon > \mathrm{kras}(f)$ implies that $\varepsilon > \gamma_{s-1}$. We can thus still apply Theorem 4.1.1 by taking $\varepsilon' := \gamma_{s-1}$ in place of ε, as long as f is not purely inseparable. Indeed,

$$v(f_\alpha - g_\alpha) \geq n\varepsilon > n\varepsilon'.$$

As a result, (4.6) remains true, that is,

$$\mathrm{NP}_{f_\alpha}(i) = \mathrm{NP}_{g_\alpha}(i) \quad \text{for} \quad t \leq i \leq n.$$

In the purely inseparable case we do not have γ_{s-1}, but the above equality still holds by Lemma 4.1.2 since $t = n$.

We can once again renumber the indices of roots of g such that (4.8) holds, that is,

$$v(\beta_t - \alpha) = \mathrm{NP}_{g_\alpha}(t-1) - \mathrm{NP}_{g_\alpha}(t).$$

Assume that f is not purely inseparable. Since $\mathrm{NP}_{f_\alpha}(t) < (n-t)\varepsilon$, (4.9) now reads

$$\mathrm{NP}_{g_\alpha}(t) = \mathrm{NP}_{f_\alpha}(t) < (n-t)\varepsilon.$$

If f is purely inseparable, the above inequalities remain weak.

We can combine our observations analogously as we did in the proof of the original theorem to obtain that for f not purely inseparable we have that

$$v(\beta_t - \alpha) = \mathrm{NP}_{g_\alpha}(t-1) - \mathrm{NP}_{g_\alpha}(t) > n\varepsilon - (n-t)\varepsilon = t\varepsilon. \tag{4.10}$$

Recall that we are assuming that $\varepsilon > \mathrm{kras}(f)$. We can thus use an analogous argument to that given in the proof to show that for every root α of f of multiplicity t there exist at least t roots of g which satisfy (4.10) and that those roots of g cannot be assigned to a root distinct from α[1]. Thus, in the case that f is not purely inseparable, the final assertion coincides with the original assertion given in Theorem 4.2.2.

However, if f is purely inseparable, then we do not have strong inequalities. Therefore we can pair up the (only) root α_k of f with n many roots β_k of g such that

$$v(\alpha_k - \beta_k) = \mathrm{NP}_{g_\alpha}(n-1) - \mathrm{NP}_{g_\alpha}(n) \geq n\varepsilon.$$

This finishes our claim.

If we now assume that the inequality in the bound for ε remains in its original weak form, then for a given root α_k of f we can still find a root β_k of g such that

$$v(\alpha_k - \beta_k) \geq t_k\varepsilon.$$

However, we do not know whether β_k can be assigned to another root $\alpha_l \neq \alpha_k$ of f. Hence, we do not know whether we can find an enumeration of roots which are pairwise close to each other.

Note that by Theorem 4.2.2, each ball $B^\circ_\varepsilon(\alpha_k)$ contains at least t_k roots of g. The following theorem gives us a more precise result for some roots of f.

[1] In fact, to prove that the t many roots β cannot be assigned to a root distinct from α, we only need that $\varepsilon \geq \mathrm{kras}(f)$. However, the condition $\varepsilon > \mathrm{kras}(f)$ is necessary to obtain strong inequality in (4.4), which in turn gives us strong inequality in (4.10), without which we could not prove the aforementioned assertion on β in the first place.

Theorem 4.2.4. *Take $f, g \in K[x]$, written as in (2.1) with f monic and $m \geq n$. Assume that for $\varepsilon \geq \max\{0, \mathrm{kras}(f)\}$ we have that*

$$v(f-g) > n\varepsilon - \deg(f-g)\gamma^*(f).$$

If $v\alpha_k > \gamma(f)$, then there are precisely t_k many roots of g (counted with multiplicity) in the ball $B_\varepsilon^\circ(\alpha_k)$.

If in addition $\varepsilon \geq (1-\frac{m}{n})\gamma(f)$, then the same holds for any α_k such that $v\alpha_k = \gamma(f)$. In this case, for $n < k \leq m$ we have that $v\beta_k < \gamma(f)$.

Proof. We employ Theorem 4.2.2 to find an enumeration of the roots of f and g such that $v(\alpha_k - \beta_k) > t_k\varepsilon \geq \varepsilon$. In particular, every ball $B_\varepsilon^\circ(\alpha_k)$ has at least t_k many roots of g. As in the previous proof, we find that $B_\varepsilon^\circ(\alpha_k) \cap B_\varepsilon^\circ(\alpha_j) = \emptyset$ if $\alpha_k \neq \alpha_j$.

As in the construction of NP_f, denote by γ_i the values of roots of f in increasing order, with γ_s being the largest. Since $\varepsilon \geq \mathrm{kras}(f)$, we must have that $\varepsilon \geq \gamma_{s-1} \geq \gamma_1 = \gamma(f)$ if f has at least two distinct values of roots, and $\varepsilon \geq \gamma_s = \gamma_1 = \gamma(f)$ if all roots of f have one value.

We will first assume that $\varepsilon \geq (1-\frac{m}{n})\gamma(f)$ and prove the claim for any root α_k of f. Since $v\alpha_k \geq \gamma_1$, the fact that $v(\alpha_k - \beta_k) > \varepsilon \geq \gamma_1$ implies that also $v\beta_k \geq \gamma_1$. By Lemma 4.1.4 combined with Theorem 3.2.2, we have that all the roots β_i, $i \in \{n+1, \ldots, m\}$, have value strictly less than γ_1. This means that there are precisely n roots of g which are eligible to be paired up with roots of f. We combine this with our previous observation to obtain that each ball $B_\varepsilon(\alpha_k)$ contains precisely t_k many roots of g.

Now take any root α_k of f such that $v\alpha_k > \gamma_1$. Since the condition $\varepsilon \geq (1-\frac{m}{n})\gamma_1$ holds if $\gamma_1 \geq 0$, we may assume that $\gamma_1 < 0$. Then $v(\alpha_k - \beta_k) \geq \varepsilon \geq 0$ implies that also $v\beta_k > \gamma_1$. Since $\varepsilon \geq \gamma_1$, the assumptions of Theorem 4.1.1 are satisfied. In particular, NP_g and NP_f coincide along the interval on which NP_f assumes slope $-\gamma_1$, that is, the interval $[k_1, n]$. If we show that on the left of the coordinate k_1, NP_g has slope strictly less than $-\gamma_1$, by Theorem 3.2.2 we will have that f and g have the same number of roots of value strictly greater than γ_1. We can then use the same argument as above to conclude that each ball $B_\varepsilon(\alpha_k)$ contains precisely t_k many roots of g.

Since there exists a root of f of value greater than $\gamma(f)$, f has at least two distinct values of roots. We therefore must have that $k_1 \neq 0$. Suppose that NP_g continues on the left of the point (k_1, va_{k_1}) with slope $-\gamma_1$. Let (i_0, vb_{i_0}), $i_0 < k_1$, be the vertex of NP_g which represents the left end of the face of NP_g that has slope $-\gamma_1$. Then

$$vb_{i_0} < \mathrm{NP}_f(i_0) \leq va_{i_0}.$$

Since $\gamma_1 < 0$ and $va_n = vb_n = 0$, this also means that $vb_{i_0} < 0$. But this implies that

$$0 > vb_{i_0} = v(b_{i_0} - a_{i_0}) \geq v(f - g) > 0,$$

which gives us a contradiction. □

The following result was presented in [1] for complete normed fields (see [1], Sect. 3.4, Proposition 1 and further results). In this dissertation, its formulation has been adapted to work with valuations of arbitrary rank that are not necessarily complete. The completeness of the field is only used in [1] to obtain a unique extension of the norm from the field to its algebraic closure. However, the statement remains true when considering any valued field (K, v) and choosing any extension of v to an algebraic closure of K. Instead of restating the original proof, we note that this theorem is a special case of Theorem 4.2.4, where $m = n$. We are also able to specify a bound in our assumptions, replacing the original epsilon-delta formulation.

Theorem 4.2.5. *Take any monic polynomial $f \in K[x]$ of degree $n \geq 1$ and let α be a root of f of multiplicity t. Choose an element $\varepsilon \in vK$ such that $\varepsilon \geq \max\{0, \operatorname{kras}(f)\}$. Assume that for a monic polynomial $g \in K[x]$ of degree n we have that*

$$v(f - g) > n\varepsilon - (n-1)\gamma^*(f).$$

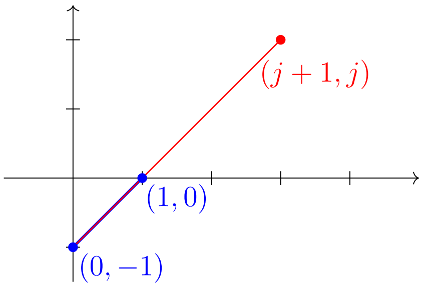
Then g has exactly t many roots (counted with multiplicities) in $B^\circ_\varepsilon(\alpha)$.

Note that as was the case of Lemma 4.1.4, the bound for $v(f - g)$ in Theorems 4.2.2 and 4.2.4 depends on both f and g. Similarly to the observation in Remark 4.1.8, this bound cannot be made independent of the polynomial g. This is illustrated in the following example.

Example 4.2.6. We claim that there exists a monic polynomial f and polynomials g_j, $j \geq 1$, such that $\{v(f - g_j) \mid j \geq 1\}$ is cofinal in vK, but for any value $\varepsilon \in vK$ and for any root α of f, the ball $B^\circ_\varepsilon(\alpha)$ contains either no roots of g_j or all roots of g_j. In particular, we will show that $B^\circ_\varepsilon(\alpha)$ contains no roots of g_j if $\varepsilon \geq 0$.

We consider the field $\mathbb{Q}$ with the 2-adic valuation v. We set $f(x) = x - \frac{1}{2}$, $g_j(x) = f(x) + 2^j x^{j+1}$. Then $v(f - g_j) = j$. Note that all roots of g_j have value -1, same as the only root $\alpha = \frac{1}{2}$ of f.

Figure 4.5: The Newton Polygons of f (blue segment) and g_j (red segment) and the corresponding vertices.



Fix any positive integer j and take $g := g_j$. To look at the values $v(\frac{1}{2} - \beta)$ for any given root β of g, we will consider the polynomial $g(x + \frac{1}{2})$. The roots of $g(x + \frac{1}{2})$ are of the form $\beta - \frac{1}{2}$, where β is any root of g. We compute:

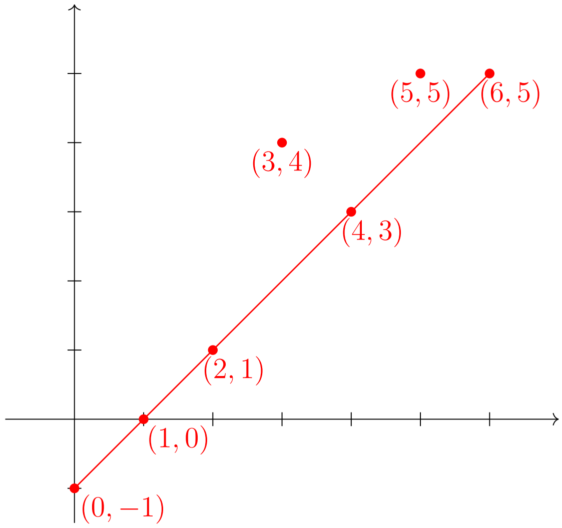
$$g\left(x + \frac{1}{2}\right) = x + 2^j \left(x + \frac{1}{2}\right)^{j+1} = x + \sum_{i=0}^{j+1} \binom{j+1}{i} 2^{i-1} x^i =: \sum_{i=0}^{j+1} b_i x^i.$$

Observe that $vb_0 = -1$ and $vb_{j+1} = j$. We claim that for $1 \le i \le j$ we have that $vb_i \ge i - 1$. Note that $b_1 = 1 + \binom{j+1}{1}$, hence $vb_1 \ge 0$. For $1 < i \le j$ we have that

$$vb_i = v\left(\binom{j+1}{i} 2^{i-1}\right) \ge v\left(2^{i-1}\right) = i - 1.$$

This means that $\mathrm{NP}_{g(x+\frac{1}{2})}$ contains precisely one face with slope 1, whose left and right endpoints are $(0, -1)$ and $(j + 1, j)$, respectively.

Figure 4.6: An example of $\mathrm{NP}_{g(x+\frac{1}{2})}$ with $j = 5$, that is, $g(x+\frac{1}{2}) = 32x^6 + 96x^5 + 120x^4 + 80x^3 + 30x^2 + 7x + \frac{1}{2}$.



Hence for any root β of g we have that $v(\frac{1}{2} - \beta) = -1$. In particular, the ball $B^\circ_\gamma(\frac{1}{2})$ contains precisely $j + 1$ roots of g if $\gamma < -1$ and it contains no roots of g for $\gamma \geq -1$.

Example 4.2.7. We will now show that the bound for $v(f - g)$ in Theorem 4.2.2 is sharp if we take $\varepsilon := \max\{0, \mathrm{kras}(f)\}$. To this end, we will again consider $\mathbb{Q}$ with the 2-adic valuation v and the polynomial $f(x) = x - \frac{1}{2}$. It has a single root $\alpha = \frac{1}{2}$ of value -1, so $\gamma^*(f) = -1$, $\mathrm{kras}(f) = -1$ and $\varepsilon = 0$. This time, we take $g_j(x) := f(x) + 2^j x^j$ for $j \in \mathbb{N}$. Then g_1 has the single root $\beta = \frac{1}{6}$, thus $v(\alpha - \beta) = -v(3) = 0$. On the other hand, $v(f - g) = v(2x) = 1$ and therefore, $n\varepsilon - \deg(f - g)\gamma^*(f) = 1 = v(f - g)$. This proves that the bound given in (4.3) is sharp.

By using the polynomials g_j defined above we can construct examples with polynomials that are arbitrarily close to f. We observe that $v(f - g_j) = j$. We fix an arbitrary $j \in \mathbb{N}$ and set $g := g_j$.

Similarly as in Example 4.2.6, we see that

$$g\left(x + \frac{1}{2}\right) = x + \sum_{i=0}^{j} \binom{j}{i} 2^i x^i =: \sum_{i=0}^{j+1} b_i x^i.$$

Then $vb_0 = vb_1 = 0$, $vb_j = j$, and $vb_i \geq i$ for $1 < i < j$. This means that the Newton Polygon of $g(x + \frac{1}{2})$ has two faces: one with slope 0 and length

1, and the other with slope $\frac{j}{j-1}$ and length $j-1$. Hence, $g(x+\frac{1}{2})$ has one root of value 0 and $j-1$ roots of value $-\frac{j}{j-1}$. If β is any root of $g(x)$, then $\beta-\frac{1}{2}$ is a root of $g(x+\frac{1}{2})$ and therefore, $v(\alpha-\beta)=v(\beta-\frac{1}{2})\leq 0$. On the other hand,

$$v(f-g)=j=n\cdot 0-j\cdot(-1)=n\varepsilon-\deg(f-g)\gamma^*(f).$$

This again shows that the strict inequality in (4.3) is necessary even when the polynomials f and g are close to each other.

We now focus on a different approach to proving root continuity theorems, which can be found in [6] and [7]. Similarly to Theorems 4.2.2 and 4.2.4, the methods presented here allow us to improve the results given in Section 2.1. To prove the following result, we will use the theory introduced in Section 3 in the particular case where $\deg f=\deg g$.

Theorem 4.2.8. *Take $\varepsilon>0$, and two monic polynomials $f,g\in K[x]$ as in (2.1) with $m=n$. Assume that*

$$v(f-g)>n\varepsilon-(n-1)\gamma^*(f). \tag{4.11}$$

Then, after suitably rearranging indices, $v(\alpha_i-\beta_i)>\varepsilon$ for every i.

Proof. Choose roots $\alpha_{i_1},\dots\alpha_{i_\ell}$ of f such that the balls $B^\circ_\varepsilon(\alpha_{i_1}),\dots,B^\circ_\varepsilon(\alpha_{i_\ell})$ are disjoint. For each $j\in\{1,\dots,\ell\}$ and $\gamma^*(f)$ given by (2.2) we have that

$$n\varepsilon-(n-1)\gamma^*(f)\geq n\varepsilon-\min\{0,(n-1)v\alpha_{i_j}\}.$$

Combined with (4.11), this shows that condition (4.2) is satisfied. Thus by Corollary 4.1.7 for $c=\alpha_{i_j}$ we have that $n_b(f,\varepsilon,\alpha_{i_j})=n_b(g,\varepsilon,\alpha_{i_j})$. We can thus enumerate the roots of g by connecting them to the roots of f that are in the same ball. □

Theorem 4.2.9. *Take $\varepsilon>0$, and two polynomials $f,g\in K[x]$ as in (2.1) with $m=n$ such that f is monic and $v(f-g)>n\varepsilon-nvf$. Then there is an enumeration of the roots of g such that $v(\alpha_i-\beta_i)>\varepsilon$ for every i.*

Proof. Recall from Lemma 2.1.5 that $vf\leq\gamma^*(f)$. Assume that $g=b_ng_0$, with g_0 a monic polynomial. We wish to show that

$$v(f-g_0)>n\varepsilon-(n-1)\gamma^*(f).$$

Since f is monic, we have that $vf\leq 0$, hence $v(f-g)>\varepsilon>0$ implies $vf=vg$. Moreover, as in the proof of Lemma 2.1.7, we see that $vb_n=0$. We compute, using the hypothesis of the theorem:

$$\begin{aligned} v(g-g_0) &= v((b_n-1)g_0)=v(b_n-1)+vg_0\geq v(f-g)+vg_0\\ &> n\varepsilon-nvf+vg-vb_n=n\varepsilon-(n-1)vf. \end{aligned}$$

As a result, we obtain that

$$v(f - g_0) \geq \min\{v(f - g), v(g - g_0)\} > n\varepsilon - (n-1)vf \geq n\varepsilon - (n-1)\gamma^*(f).$$

Applying Theorem 4.2.8 to f and g_0 in place of g yields the required result. □

Remark 4.2.10. Note that if $\deg f = \deg g = n$, then $\deg(f - g) \leq n - 1$. Hence under the additional assumption that $\varepsilon \geq \operatorname{kras}(f)$, Theorem 4.2.2 generalizes Theorem 4.2.8. Similarly, since $\gamma^*(f) \leq vf \leq 0$ and $\deg(f - g) \leq n$ for f and g of degree n, with the same additional assumption our theorem generalizes Theorem 4.2.9. However, the latter results are useful if $v(f - g)$ is a small positive value. Consider $f(x) = x^2 - 16$ and $g(x) = x^2 - 4$ in the field $\mathbb{Q}$ with the 2-adic valuation. In this case, Theorem 4.2.2 does not work since $v(f - g) < \operatorname{kras}(f)$, but taking $\varepsilon = \frac{1}{2}$ allows us to use Theorem 4.2.8.

4.3 The separant and the error function

In the previous section we have stated an alteration of [2, Theorem 1]. In this section we will have a look at this theorem, along with other selected results from that source.

Let $f \in K[x]$ be a monic polynomial as in (2.1). *The separant of f* is the following value:

$$\mathscr{S}_f := \max\{v(f'(\alpha_i)) + v(\alpha_i - \alpha_j) \mid i, j \in \{1, \ldots, n\} \wedge i \neq j\}.$$

We see that $\mathscr{S}_f < \infty$ if and only if f is separable. Thus the following theorem implicitly assumes the separability of f. For comparison results between this theorem and other results, see Remarks 4.3.2 and 4.3.3.

Theorem 4.3.1. *Let (K, v) be a valued field and let f and g be two polynomials in $\mathcal{O}_K[x]$, written as in (2.1) with $m \geq n$. Assume that f is monic and that $v(f - g) > \mathscr{S}_f$. Then, after a suitable renumbering of indices, we have that $v(\alpha_k - \beta_k) > v(\alpha_k - \alpha_j)$ for all $j, k \in \{1, \ldots, n\}$ such that $j \neq k$. Moreover, all the roots β_k, $k \in \{1, \ldots, n\}$ are pairwise distinct.*

If in addition (K, v) is Henselian and $m = n$, then for each k we have that $K(\alpha_k) = K(\beta_k)$.

Proof. Write $N := \{1, \ldots, n\}$ and fix $k \in N$. Take $\alpha := \alpha_k$ and consider the polynomials $f_\alpha(x) = f(x + \alpha)$, $g_\alpha(x) = g(x + \alpha)$. Denote by a'_i and b'_i the respective coefficients of f_α and g_α. Since 0 is a root of f_α, we have that $\mathrm{NP}_{f_\alpha}(0) = va'_0 = \infty$. As in Equation (4.4) with $t = 1$, we prove that

$$\mathrm{NP}_{f_\alpha}(1) = \sum_{j \in N, j \neq k} v(\alpha - \alpha_j) = vf'(\alpha). \tag{4.12}$$

Since f is monic and has integral coefficients, we have that $v\alpha \geq 0$ by part (a) of Lemma 3.2.4, and so by Lemma 1.6.3, $v(f_\alpha - g_\alpha) \geq v(f-g) > \mathscr{S}_f$. Therefore,

$$\mathrm{NP}_{g_\alpha}(0) = vb_0' = v(b_0' - a_0') \geq v(f_\alpha - g_\alpha) > \mathscr{S}_f. \tag{4.13}$$

We will now prove that $\mathrm{NP}_{f_\alpha}(l) = \mathrm{NP}_{g_\alpha}(l)$ for $l \in N$.

First, take $l > 0$ such that (l, va_l') is a vertex of NP_{f_α}. Consider the sequence of all roots of f_α (that is, elements of the form $\alpha_i - \alpha$), ordered so that their values are non-decreasing. Let the set $I \subseteq N$ consist of the indices i such that $(\alpha_i - \alpha)$ are the first $n-l$ elements in that sequence. Then by (3.1) we have that $va_l' = \sum_{i\in I} v(\alpha_i - \alpha)$. Since all roots of f are integral, for all i we have that $v(\alpha_i - \alpha) \geq 0$. Therefore,

$$\left.\begin{array}{rcl} va_l' &=& \sum_{i\in I} v(\alpha_i - \alpha) \leq \sum_{j\in N, j\neq k} v(\alpha - \alpha_j) \\ &\leq& \max\left\{\sum_{t\in N, t\neq i} v(\alpha_i - \alpha_t) + v(\alpha_i - \alpha_j) \mid i,j \in N \wedge i \neq j\right\} \\ &=& \max\left\{vf'(\alpha_i) + v(\alpha_i - \alpha_j) \mid i,j \in N \wedge i \neq j\right\} = \mathscr{S}_f. \end{array}\right\} \tag{4.14}$$

On the other hand, $v(a_l' - b_l') > \mathscr{S}_f$, thus $va_l' = vb_l'$.

Take now $l > 0$ such that (l, va_l') is not a vertex of NP_{f_α}. Then $va_l' \geq \mathrm{NP}_{f_\alpha}(l)$. Since f is separable, 0 is a simple root of f_α. Hence, 1 and n are respectively the first and the last coordinate at which NP_{f_α} is finite. Thus both $(1, va_1')$ and (n, va_n') are vertices of NP_{f_α}. Therefore, we can use inequality (4.14) for $l = 1$ and $l = n$ to see that both va_1' and va_n' have value at most $\mathscr{S}_f$. Since NP_{f_α} is upward convex, we must have that $\mathrm{NP}_{f_\alpha}(l) \leq \max\{va_1', va_n'\} \leq \mathscr{S}_f$. We combine our observations to obtain

$$vb_l' \geq \min\{va_l', v(a_l' - b_l')\} \geq \min\{\mathrm{NP}_{f_\alpha}(l), \mathscr{S}_f\} = \mathrm{NP}_{f_\alpha}(l).$$

We have now shown that $va_l' = vb_l'$ for l such that (l, va_l') is a vertex of NP_f, and that $vb_l' \geq \mathrm{NP}_{f_\alpha}(l)$ for every other l.

Since $v(f_\alpha - g_\alpha) > 0$ and f_α has integral roots, by Lemma 4.1.4, (n, vb_n') is a vertex of NP_{g_α}. Let $\ell \geq 1$ be the minimal index such that (ℓ, vb_ℓ') is a vertex of NP_{g_α}. Then the Newton Polygons of f and g coincide along the interval $[\ell, n]$. Therefore, if we show that $\ell = 1$, then

$$\mathrm{NP}_{f_\alpha}(l) = \mathrm{NP}_{g_\alpha}(l) \text{ for } l \in N. \tag{4.15}$$

If $n = 1$, then the claim is satisfied, thus assume that $n > 1$. Write

$$\gamma := \max\{v(\alpha_j - \alpha) \mid j \in N, j \neq k\}.$$

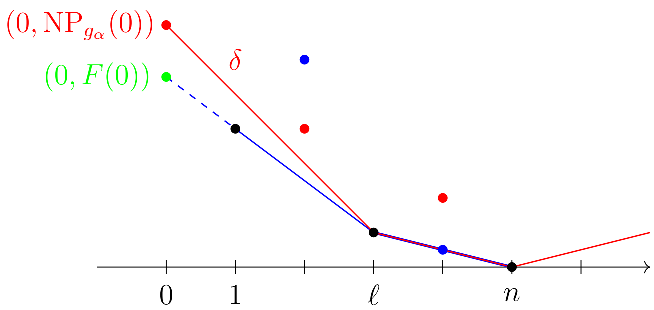
Note that $-\gamma$ is the leftmost amongst the slopes of NP_{f_α} which are not equal to $-\infty$. Define a function $F : [0, n] \to \Gamma$ such that $F(x) = \mathrm{NP}_{f_\alpha}(x)$ for $x \geq 1$, and on the interval $[0, 1]$, F is given by the line going through the point $(1, \mathrm{NP}_{f_\alpha}(1))$ with slope $-\gamma$. In other words, we continue on the left of the point $(1, \mathrm{NP}_{f_\alpha}(1))$ with the same slope as we have approached this point from the right. Note that F is a piecewise linear upper convex function on the interval $[0, n]$.

We combine equations (4.12) and (4.13), together with the definition of the separant, to obtain that

$$\mathrm{NP}_{g_\alpha}(0) > \mathscr{S}_f = \max_{i \neq j}\{f'(\alpha_i) + v(\alpha_i - \alpha_j)\} \geq \mathrm{NP}_{f_\alpha}(1) + \gamma = F(0).$$

Recall that ℓ is the minimal index in N such that (ℓ, vb'_ℓ) is a vertex of NP_{g_α}. Suppose, aiming for a contradiction, that $\ell \neq 1$. The leftmost face of NP_{g_α} is given by a linear function connecting $(0, \mathrm{NP}_{g_\alpha}(0))$ and $(\ell, \mathrm{NP}_{g_\alpha}(\ell))$ on $[0, \ell]$. Denote this function by δ.

Figure 4.7: The Newton Polygons of f_α (blue segments) and g_α (red segments), together with the function F which coincides with NP_{f_α} along $[1, n]$ and is given by the dashed blue segment along $[0, 1]$. The leftmost face of NP_{g_α} is given by the linear function δ. The blue points correspond to the points (i, va'_i), the red points represent (i, vb'_i), and the black points are located in places where the two of the above coincide.



We have that

$$F(\ell) = \mathrm{NP}_{f_\alpha}(\ell) = \mathrm{NP}_{g_\alpha}(\ell) = \delta(\ell) \quad \text{and} \quad F(0) < \delta(0).$$

Since F is upward convex and δ is linear, for all $x \in [0, \ell)$ we have that $F(x) < \delta(x)$. Since $\mathrm{NP}_{f_\alpha}(0) = \infty$ and $\mathrm{NP}_{f_\alpha}(1) \neq \infty$, $(1, va'_1)$ is a vertex of NP_{f_α} and thus by what we have shown, $va'_1 = vb'_1$. Hence also the point

$$(1, F(1)) = (1, \mathrm{NP}_{f_\alpha}(1)) = (1, va'_1) = (1, vb'_1)$$

lies strictly below the function δ. But δ was a function defining the face of NP_{g_α} on the real interval $[0, \ell]$. As a result, $(1, vb_1')$ lies below NP_{g_α}, contradiction. This proves that $\ell = 1$ and so (4.15) holds.

Recall that we are considering the root $\alpha = \alpha_k$ of f. For the time being, we write the indices of roots of g in such a way that

$$v(\beta_k - \alpha) \geq v(\beta_i - \alpha)$$

for all $i \in \{1, \ldots, m\}$. Then $\beta_k - \alpha$ is a root of g_α with maximal value, thus

$$v(\beta_k - \alpha) = \mathrm{NP}_{g_\alpha}(0) - \mathrm{NP}_{g_\alpha}(1).$$

As a result, we obtain that

$$v(\beta_k - \alpha) = \mathrm{NP}_{g_\alpha}(0) - \mathrm{NP}_{g_\alpha}(1) > \mathscr{S}_f - vf'(\alpha_k) \geq v(\alpha_k - \alpha_j) \text{ for all } j \neq k,$$

where the first inequality follows from (4.12), (4.13) and the fact that

$$\mathrm{NP}_{g_\alpha}(1) = \mathrm{NP}_{f_\alpha}(1).$$

We have thus shown that for α_k there exists a root β_k such that

$$v(\beta_k - \alpha_k) > v(\alpha_k - \alpha_j) \text{ for all } j, k \in N \text{ such that } j \neq k. \tag{4.16}$$

We claim that if we repeat this construction for another root α_l with $l \neq k$, then the resulting paired root β_l will not be equal to β_k. Indeed, suppose that $\beta_l = \beta_k$. Then

$$v(\alpha_k - \alpha_l) \geq \min\{v(\alpha_k - \beta_k), v(\alpha_l - \beta_k)\} > v(\alpha_k - \alpha_l),$$

a contradiction. We have that for every α_k there exists a root β_k which satisfies our claim. Moreover, this root β_k cannot be assigned to a root different from α_k. We can thus renumber the roots of g such that $v(\alpha_k - \beta_k) > v(\alpha_k - \alpha_j)$ for all $k, j \in \{1, \ldots, n\}$ with $j \neq k$, assigning indices from $\{n+1, \ldots, m\}$ to the roots of g which were not chosen to be paired with any root of f. Since f is separable, also the roots $\beta_1, \ldots, \beta_n$ of g must be pairwise distinct.

Assume now that $m = n$ and that K is Henselian. Since f has now precisely as many roots as g, the choice of the root β_k for α_k is unique. As a result, the claim (4.16) does not hold for any root β_l, $l \neq k$. Hence, there exists an index $j \neq k$ such that

$$v(\alpha_k - \beta_l) \leq v(\alpha_k - \alpha_j) < v(\alpha_k - \beta_k).$$

Consequently,

$$v(\beta_k - \beta_l) = \min\{v(\alpha_k - \beta_k), v(\alpha_k - \beta_l)\} = v(\alpha_k - \beta_l) < v(\alpha_k - \beta_k).$$

Thus for each k we have that

$$v(\alpha_k - \beta_k) > \mathrm{kras}_K(\alpha_k) \text{ and } v(\alpha_k - \beta_k) > \mathrm{kras}_K(\beta_k).$$

The last assertion of the theorem then follows from Lemma 4.2.1.

□

Remark 4.3.2. The original formulation of the above theorem in [2] does not explicitly state that $v(\alpha_k - \beta_k) > v(\alpha_k - \alpha_j)$ for every $k, j \in \{1, \dots, n\}$, $k \neq j$. Instead, it assumes that K is Henselian and that $m = n$, claiming that $K(\alpha_i) = K(\beta_i)$ for every i. As was the case for Theorem 4.2.2, we do not need Henselianity to prove that the respective roots are close to each other. Thus the formulation of the above theorem gives us slightly more information than the one given in [2].

Remark 4.3.3. We note that the formulation and the proof of Theorem 4.3.1 are similar to those of Theorem 4.2.2. However, in their current forms, neither is a generalization of the other. Indeed, on the one hand, the bound $\mathscr{S}_f$ is more precise than the one given in (4.3). On the other hand, Theorem 4.3.1 assumes that the polynomials in question have integral coefficients and that f is separable (which is hidden in the assumption that $\mathscr{S}_f \neq \infty$). In order to be able to consider polynomials that need not have integral coefficients and still be able to apply Lemma 1.6.3, we need to consider the additional value $\deg(f-g)\gamma^*(f)$ in the bound for $v(f-g)$. This justifies the summand $\deg(f-g)\gamma^*(f)$ in (4.3). Note that the separant $\mathscr{S}_f$ is a sum of n many summands, each of which is of the form $v(\alpha_i - \alpha_j)$ for some i, j (cf. (4.14)). In the definition of ε in Theorem 4.2.2, each of those summands is replaced by the maximal value, that is, $\mathrm{kras}(f)$. Accounting again for the possible negative value of some roots, we then take $\varepsilon \geq \max\{0, \mathrm{kras}(f)\}$. While this replacement seems minor at first, it is necessary to be able to employ Theorem 4.1.1 in the proof of Theorem 4.2.2, which required that $v(f_\alpha - g_\alpha) > n\varepsilon$. Proving that the Newton Polygons of f and g coincide was thus more involved in the proof of Theorem 4.3.1 specifically for the reason that the bound $\mathscr{S}_f$ is more precise than the bound given in (4.3).

For the next result, we will employ the following notion.

Let I be an ordered set such that the order type of I equals the principal rank of vK. Consider the Hahn product $\Gamma := \mathbf{H}_{i\in I}\mathbb{R}$ endowed with the lexicographic order. The *error function* of a root α of f is the map

$$\Phi : \Gamma \cup \{\infty\} \to \Gamma \cup \{\infty\}$$

given by

$$\Phi(x) = \sum_{i=1}^{n} \min\{x, v(\alpha - \alpha_i)\}.$$

If we wish to specify the root, we will write Φ_α in place of Φ.

Observe that Φ is strictly increasing. Moreover, Φ is given by linear functions $\ell_j(x) = a_j x + \gamma_j$ for some $a_j \in \mathbb{N}_0$ and $\gamma_j \in \Gamma$. We can thus define the slopes of Φ to be the numbers a_j. Then the vertices of Φ are located at $x = v(\alpha - \alpha_k)$ for some numbers $k \in \{1, \ldots, n\}$. The slopes of Φ are decreasing, with the first slope equal to n, and the last slope equal to the multiplicity of α. Moreover, $\Phi(\infty) = \{\infty\}$ and $\Phi(\Gamma) = \Gamma$ since Γ is divisible. In particular, Φ is a bijection. Note that $\Phi^{-1}(x) > 0$ if $x > 0$, since

$$n\Phi^{-1}(x) \geq \sum_{i=1}^{n} \min\{\Phi^{-1}(x), v(\alpha - \alpha_i)\} = \Phi(\Phi^{-1}(x)) = x > 0.$$

Moreover, if $f \in \mathcal{O}_K[x]$ is monic, then $\Phi^{-1}(x) < 0$ if $x < 0$, since in this case we have that $\Phi(x) = nx$. In particular, in this case we have that $\Phi^{-1}(x) = 0$ if and only if $x = 0$.

Let α_1 and α_2 be two roots of f. We claim that

$$\Phi_{\alpha_1}(x) = \Phi_{\alpha_2}(x) \text{ for all } x \leq v(\alpha_1 - \alpha_2). \tag{4.17}$$

To prove this, we will show that for every $i \in \{1, \ldots, n\}$ we have that $x \leq v(\alpha_1 - \alpha_i)$ if and only if $x \leq v(\alpha_2 - \alpha_i)$. Indeed, suppose that $x > v(\alpha_1 - \alpha_i)$ and $x \leq v(\alpha_2 - \alpha_i)$, then

$$x \leq v(\alpha_1 - \alpha_2) = v(\alpha_1 - \alpha_i) < x,$$

which gives us a contradiction. The opposite implication is proved analogously. This proves (4.17).

Theorem 4.3.4. *Let f and g be monic polynomials in $\mathcal{O}_K[x]$ as in (2.1) with $m = n$. Then there exists an enumeration of indices such that*

$$v(\alpha_k - \beta_k) \geq \Phi_{\alpha_k}^{-1}(v(f - g)) \textit{ for each } k \in \{1, \ldots, n\},$$

where Φ_{α_k} is the error function of the root α_k of f.

Proof. Fix any root α_k of f. Write $\alpha := \alpha_k$ and define

$$\rho := \Phi_\alpha^{-1}(v(f - g)).$$

We will first show that the claim holds trivially unless we assume that $0 < v(f - g) < \infty$. Indeed, in the case of $v(f - g) = \infty$ there is nothing to show. Since f and g are polynomials in $\mathcal{O}_K[x]$, we have that $v(f-g) \geq 0$. Assume that $v(f-g) = 0$. We once again use the fact that $f \in \mathcal{O}_K$ and f is monic, thus by our observations before the theorem, $\Phi_\alpha^{-1}(0) = 0$. But then the claim of the theorem states that $v(\alpha_k - \beta_k) \geq 0$, which is always true since $v\alpha_k \geq 0$ and $v\beta_k \geq 0$ (cf. part (a) of Lemma 3.2.4). Thus we may now assume that $0 < v(f-g) < \infty$.

Recall that $\Phi_\alpha^{-1}(x) > 0$ if $x > 0$. Therefore we also have that $0 < \rho < \infty$. We wish to show that f and g have the same number of roots (counted with multiplicity) in the ball

$$B_\rho(\alpha) = \{c \in \Gamma \mid v(c-\alpha) \geq \rho\}.$$

Once this is shown for an arbitrary root α, we will be able to pair up the corresponding roots of f and g which are in the same ball, since ultrametric balls are either disjoint or comparable by inclusion. This pairing will then satisfy our claim.

Let μ be the number of indices i with $v(\alpha_i - \alpha) < \rho$. We wish to show that μ is also the number of indices j such that $v(\beta_j - \alpha) < \rho$.

Write

$$f_\alpha(x) = \sum_{i=0}^{n} a'_i x^i$$

and consider the Newton Polygon NP_{f_α}. The slopes of NP_{f_α} are of the form $-v(\alpha_i - \alpha)$ in increasing order. Therefore, $\mathrm{NP}_{f_\alpha}(i-1) - \mathrm{NP}_{f_\alpha}(i) < \rho$ for $n - \mu < i \leq n$, and $\mathrm{NP}_{f_\alpha}(i-1) - \mathrm{NP}_{f_\alpha}(i) \geq \rho$ for $i \leq n-\mu$. Consider the line ℓ going through the point $(n-\mu, va'_{n-\mu})$ with slope $-\rho$, that is, $\ell(i) = (n-\mu-i)\rho + va'_{n-\mu}$. Then:

$$\mathrm{NP}_{f_\alpha}(i) \begin{cases} \geq \ell(i), & i < n-\mu \\ = \ell(i), & i = n-\mu \\ > \ell(i), & i > n-\mu. \end{cases}$$

Then the proof will be finished if we show the same for the Newton Polygon NP_{g_α} of the polynomial

$$g_\alpha = \sum_{i=1}^{n} b'_i x^i.$$

Recall that $(n-\mu)$ is the number of roots α_i such that $v(\alpha_i - \alpha) \geq \rho$. Moreover, since $a'_{n-\mu}$ gives us a vertex of NP_{f_α}, by (3.1) $va'_{n-\mu}$ is equal to the sum of the smallest μ many values of roots of f_α. Therefore, if we write

$$J := \{j \in \{1,\ldots,n\} \mid v(\alpha_j - \alpha) \geq \rho\}, \quad I := \{1,\ldots,n\} \setminus J,$$

then

$$\begin{aligned}\ell(0) &= (n-\mu)\rho + va'_{n-\mu} = \sum_{j\in J}\rho + \sum_{i\in I} v(\alpha_i-\alpha) \\ &= \sum_{i=1}^{n} \min\{\rho, v(\alpha_i-\alpha)\} = \Phi_\alpha(\rho) = v(f-g) = v(f_\alpha - g_\alpha),\end{aligned}$$

where the last equality follows from part (a) of Lemma 3.2.4 and Lemma 1.6.1. Note that $\mu < n$ by definition. We claim that:

$$\mathrm{NP}_{g_\alpha}(i) \begin{cases} \geq \ell(i), & i < n-\mu \\ = \mathrm{NP}_{f_\alpha}(i), & i \geq n-\mu. \end{cases}$$

For $i < n-\mu$ we have that $va'_i \geq \ell(i)$ and

$$v(a'_i - b'_i) \geq v(f_\alpha - g_\alpha) = \ell(0) \geq \ell(i).$$

This implies that $vb'_i \geq \ell(i)$.

For the case of $i \geq n-\mu$ we will first take i such that (i, va'_i) is a vertex of NP_{f_α}. We know that:

$$\ell(0) = v(f_\alpha - g_\alpha) > 0 = va'_n, \quad \text{and} \quad \ell(0) > \ell(n-\mu) = va'_{n-\mu}. \tag{4.18}$$

If $va'_{n-\mu} = 0 = va'_n$, then there exist no indices $i \geq n-\mu$ which give us vertices of NP_{f_α}. If $va'_{n-\mu} > 0 = va'_n$, then for every $i \geq n-\mu$ we have that

$$va'_i \leq va'_{n-\mu} < \ell(0) = v(f_\alpha - g_\alpha) \leq v(b'_i - a'_i).$$

This means that for every index $i \geq n-\mu$ for which (i, va'_i) is a vertex of NP_{f_α} we have that $va'_i = vb'_i$.

Now consider $i \geq n-\mu$ which does not give us a vertex of NP_{f_α}. We know that $va'_i \geq \mathrm{NP}_{f_\alpha}(i)$ and that

$$\mathrm{NP}_{f_\alpha}(i) \leq \max\{va'_{n-\mu}, va'_n\} < v(f_\alpha - g_\alpha),$$

where the last inequality follows from (4.18). Hence,

$$vb'_i \geq \min\{va'_i, v(a'_i - b'_i)\} \geq \min\{\mathrm{NP}_{f_\alpha}(i), v(f_\alpha - g_\alpha)\} = \mathrm{NP}_{f_\alpha}(i).$$

We have now shown that $va'_i = vb'_i$ for each index $i \geq n-\mu$ that gives us a vertex of NP_{f_α}, and that $vb'_i \geq \mathrm{NP}_{f_\alpha}(i)$ for other indices $i \geq n-\mu$. On the one hand, recall that for $i < n-\mu$, the point b'_i lies on or above the line ℓ. On the other hand, the slope located on the right of the point $(n-\mu, va'_{n-\mu})$ is strictly greater than the slope of ℓ. This means that the point $(n-\mu, vb'_{n-\mu})$ must also be a vertex of NP_{g_α} and therefore

$$\mathrm{NP}_{f_\alpha}(i) = \mathrm{NP}_{g_\alpha}(i) \text{ for } i \geq n-\mu.$$

This finishes the proof. □

Remark 4.3.5. Take f and g as in Theorem 4.3.4. Assume that α_k is a root of f such that $\alpha_k v$ is a root of fv of multiplicity μ. This means that $v(\alpha_k - \alpha_i) > 0$ for precisely μ values of i, and $v(\alpha_k - \alpha_i) = 0$ for the remaining indices i. By Theorem 4.3.4 we have that $v(\alpha_k - \beta_k) \geq \Phi_\alpha^{-1}(v(f-g))$. This implies that

$$v(f-g) \leq \Phi_\alpha(v(\alpha_k - \beta_k)) = \sum_{i=1}^{n} \min\{v(\alpha_k - \beta_k), v(\alpha_k - \alpha_i)\} \leq \mu v(\alpha_k - \beta_k).$$

Thus, $v(\alpha_k - \beta_k) \geq \frac{v(f-g)}{\mu}$. In particular, if $v(f-g) > 0$, then for all k we have that

$$v(\alpha_k - \beta_k) \geq \frac{v(f-g)}{n}.$$

This is a form of root continuity which resembles the classical results such as Theorem 2.1.3.

We have seen that Theorem 4.2.2 is an alteration of Theorem 4.3.1 which does not require the polynomials to have integral coefficients, at the cost of exchanging the separant with another bound.

We claim that the assumption on f and g having integral coefficients is necessary for Theorem 4.3.4 to remain true with the original error function. It remains an open question whether one can replace the error function with another function such that the resulting theorem will not require the polynomials to be in $\mathcal{O}_K[x]$.

We will show examples of monic polynomials f and g of degree n such that there exists a root α of f such that for every root β of g such that

$$v(\alpha - \beta) < \Phi_\alpha^{-1}(v(f-g)).$$

Note that the proof of Theorem 4.3.4 assumes without loss of generality that $v(f-g) > 0$ since otherwise the claim trivially holds for $f, g \in \mathcal{O}_K[x]$. We will give one counterexample each for $v(f-g)$ being positive, negative and equal to zero. In each of the cases we assume that we are working over the algebraic closure of $\mathbb{Q}$ with any extension of the 2-adic valuation.

Example 4.3.6. The example for $v(f-g) < 0$ is as follows: we take

$$f(x) = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{4}\right) = x^2 - \frac{3}{4}x + \frac{1}{8},$$

$$g(x) = (x-1)^2 = x^2 - 2x + 1.$$

We have that $v(f-g) = -3$. The error function Φ is the same for both roots of f. We compute:

$$\Phi(x) = x + \min\left\{x, v\left(\frac{1}{2} - \frac{1}{4}\right)\right\} = x + \min\{x, -2\} = \begin{cases} 2x, & x \leq -2 \\ x-2, & x \geq -2. \end{cases}$$

Then $\Phi^{-1}(v(f-g)) = -1$. But we have that

$$v\left(\frac{1}{4} - 1\right) = v\left(-\frac{3}{4}\right) = -2 < -1.$$

Example 4.3.7. For the counterexample with $v(f-g) = 0$ we take

$$f(x) = \left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) = x^2 - \frac{1}{4},$$

$$g(x) = \left(x - \frac{1+\sqrt{2}}{2}\right)\left(x - \frac{1-\sqrt{2}}{2}\right) = x^2 - x - \frac{1}{4}.$$

Then

$$\Phi(x) = x + \min\{x, v(-1)\} = x + \min\{x, 0\} = \begin{cases} 2x, & x \leq 0 \\ x, & x \geq 0. \end{cases}$$

Note that $\Phi^{-1}(0) = 0$. We have that

$$v\left(\frac{1}{2} - \frac{1+\sqrt{2}}{2}\right) = v\frac{\sqrt{2}}{2} = \frac{1}{2} - 1 = -\frac{1}{2} < 0.$$

$$v\left(-\frac{1}{2} - \frac{1+\sqrt{2}}{2}\right) = v\left(\frac{2+\sqrt{2}}{2}\right) = v\left(1 + \frac{\sqrt{2}}{2}\right) = -\frac{1}{2} < 0.$$

Example 4.3.8. For the counterexample where $v(f-g) > 0$, we take

$$f(x) = \left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) = x^2 - \frac{1}{4},$$

$$g(x) = \left(x - \frac{2-\sqrt{5}}{2}\right)\left(x - \frac{2+\sqrt{5}}{2}\right) = x^2 - 2x - \frac{1}{4}.$$

The error function is as in the previous example and $\Phi^{-1}(v(f-g)) = \Phi^{-1}(1) = 1$. Note that $v(1+\sqrt{5}) = v(1-\sqrt{5}) = 1$ because $v((1+\sqrt{5})(1-\sqrt{5})) = 2$ and $v((1+\sqrt{5}) + (1-\sqrt{5})) = 1$. We thus can compute:

$$v\left(\frac{1}{2} - \frac{2+\sqrt{5}}{2}\right) = v\left(\frac{1+\sqrt{5}}{-2}\right) = 0 < 1.$$

$$v\left(\frac{1}{2} - \frac{2-\sqrt{5}}{2}\right) = v\left(\frac{1-\sqrt{5}}{2}\right) = 0 < 1.$$

Chapter 5

Continuity of roots and poles for rational functions

5.1 Extending the Gauß valuation to $K(x)$

In this chapter, we study possible generalizations of root continuity theorems on the polynomial ring $K[x]$ to the rational function field $K(x)$. Note that the Gauß valuation we have used on $K[x]$ was a tool to be able to approximate a polynomial by means of approximating its coefficients. For our applications (see Chapter 6), it is useful to be able to find a polynomial whose coefficients are close enough to those of a fixed polynomial which we are considering. To have a similar look at the field $K(x)$, we may consider a "naive" approach. That is, two rational functions

$$F(x) := \frac{f(x)}{\hat{f}(x)} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{\hat{n}} \hat{a}_i x^i}, \quad G(x) := \frac{g(x)}{\hat{g}(x)} = \frac{\sum_{i=0}^{m} b_i x^i}{\sum_{i=0}^{\hat{m}} \hat{b}_i x^i}$$

will be "close to each other" if the respective values $v(a_i - b_i)$ and $v(\hat{a}_i - \hat{b}_i)$ are "large". However, in this setting, we can make any two rational functions arbitrarily close to each other by multiplying the numerators and denominators by an element in K of sufficiently large value. To avoid this situation, we will fix the representation of the functions in $K(x)$. Any rational function $H(x) = \frac{h(x)}{\hat{h}(x)} \in K(x)$ can be written uniquely in such a way that h and $\hat{h}$ are coprime, and such that $\hat{h}$ is monic. **We assume that each rational function considered in this section is represented this way.**

With this convention, we can make the "naive" statement more precise. We define a mapping $u : K(x) \times K(x) \to vK$ as follows:

$$u(F, G) := \min\{v(f - g), v(\hat{f} - \hat{g})\}.$$

Since we are now considering only one representation of each rational function, the polynomials $f, g, \hat{f}, \hat{g}$ are uniquely determined by F and G. Therefore, this mapping is well-defined. We will then say that two rational functions $F, G \in K(x)$ are close to each other, if the value $u(F,G)$ is large.

Note that there is a second natural way to define closeness of two rational functions. Given the Gauß valuation on $K[x]$, we extend it to the valued field $(K(x), v)$ by setting

$$v\left(\frac{f}{\hat{f}}\right) := vf - v\hat{f}.$$

This valuation gives us an ultrametric $w : K(x) \times K(x) \to vK$ given by

$$w(F,G) := v(F-G) = v(f\hat{g} - g\hat{f}) - v(\hat{f}\hat{g}).$$

Note that the mapping w does not depend on the representation of the functions $f, \hat{f}, g, \hat{g}$. Indeed, multiplying each of the numerators and denominators by a term will appear in $v(f\hat{g} - g\hat{f})$ and in $v(\hat{f}\hat{g})$, hence it will cancel once we calculate $v(F-G)$.

Proposition 5.1.1. *The mapping u is an ultrametric on $K(x)$ which extends the ultrametric given by the Gauß valuation v on $K[x]$.*

Proof. We have that $u(F,G) = \infty$ if and only if $f = g$ and $\hat{f} = \hat{g}$. Since in our setting the polynomials $f, \hat{f}, g, \hat{g}$ are uniquely determined by the functions F and G, this happens if and only if $F = G$. Moreover, $u(F,G) = u(G,F)$ follows directly from the definition of u.

Consider now a third rational function $H = \frac{h}{\hat{h}}$. Then

$$\begin{aligned} u(F,H) &= \min\{v(f-h), v(\hat{f}-\hat{h})\} \\ &\geq \min\{\min\{v(f-g), v(g-h)\}, \min\{v(\hat{f}-\hat{g}), v(\hat{g}-\hat{h})\}\} \\ &= \min\{\min\{v(f-g), v(\hat{f}-\hat{g})\}, \min\{v(g-h), v(\hat{g}-\hat{h})\}\} \\ &= \min\{u(F,G), u(G,H)\}. \end{aligned}$$

Hence, u is an ultrametric on $K(x)$. Since each pair of polynomials $f, g \in K[x]$ has a unique representation of the form $\frac{f}{1}$ and $\frac{g}{1}$, we have that $u(f,g) = v(f-g)$. Therefore, u extends the ultrametric given by the Gauß valuation on $K[x]$. □

We will now have a closer look at the ultrametrics u and w on $K(x)$.

Proposition 5.1.2. *Assume that $(G_j)_{j\in I}$ is a net of rational functions converging to $F \in K(x)$ with respect to the ultrametric u. Then $(G_j)_{j\in I} \to F$ with respect to w. However, the converse does not hold in general.*

Proof. Fix $F = \frac{f}{\hat{f}} \in K(x)$ and a net of rational functions $(G_j)_{j\in I} := \left(\frac{g_j}{\hat{g}_j}\right)_{j\in I}$ for some directed set I. Assume that this net converges to F with respect to u. In other words, for every $\delta \in vK$ there exists $j_0 \in I$ such that for every $j \in I$, $j \geq j_0$,

$$v(f - g_j) > \delta \quad \text{and} \quad v(\hat{f} - \hat{g}_j) > \delta.$$

Fix any $\varepsilon \in vK$. We wish to show that from some point on, $w(F, G_j) > \varepsilon$. We choose $j_0 \in I$ such that for all $j \in I$, $j \geq j_0$, we have that $u(F, G) > \delta := \varepsilon - \min\{vf, v\hat{f}\}$. Then:

$$\begin{aligned}
w(F, G_j) &= v\left(\frac{f}{\hat{f}} - \frac{g_j}{\hat{g}_j}\right) \\
&= v(f\hat{g}_j - g_j\hat{f}) - v(\hat{f}\hat{g}_j) \\
&\geq v(f\hat{g}_j - g_j\hat{f}) = v(f\hat{g}_j - f\hat{f} + f\hat{f} - g_j\hat{f}) \\
&\geq \min\{v(f\hat{g}_j - f\hat{f}), v(f\hat{f} - g_j\hat{f})\} \\
&= \min\{vf + v(\hat{f} - \hat{g}_j), v\hat{f} + v(f - g_j)\} \\
&> \min\{vf + \varepsilon - \min\{vf, v\hat{f}\}, v\hat{f} + \varepsilon - \min\{vf, v\hat{f}\}\} \\
&\geq \varepsilon.
\end{aligned}$$

Hence, if $(G_j)_{j\in I} \to F$ with respect to u, then the same holds with respect to w. To prove that the converse does not hold, we will look at the following example.

Example 5.1.3. Consider $\mathbb{Q}$ with the 2-adic valuation v, take I to be the set of positive integers and set $\alpha := \frac{1}{2}$. We take:

$$F(x) := \frac{1}{1}, \quad G_j(x) := \frac{\alpha^j}{x + \alpha^j}.$$

We claim that $w(F, G_j) = j$, but $u(F, G_j) = v(f - g_j) = v(\hat{f} - \hat{g}_j) = -j$. Indeed,

$$\begin{aligned}
w(F, G_j) &= v\left(\frac{1}{1} - \frac{\alpha^j}{x + \alpha^j}\right) = v\left(\frac{x + \alpha^j - \alpha^j}{x + \alpha^j}\right) \\
&= vx - v\left(x + \alpha^j\right) = 0 - (-j) = j.
\end{aligned}$$

On the other hand,

$$v\left(\alpha^j - 1\right) = -j \quad \text{and} \quad v\left(x + \alpha^j - 1\right) = -j.$$

Hence we have found a net of polynomials convergent to F with respect to w, but not convergent to F with respect to u.

□

Note that in particular, for the rational functions in Example 5.1.3 we have that

$$u(F, G_j) \neq w(F, G_j) = v(F - G_j).$$

We know that the ultrametric $u|_{K[x]}$ on $K[x]$ comes from the Gauß valuation v. The extension of v from $K[x]$ to $K(x)$ is unique, since for every $H = \frac{h}{\hat{h}} \in K(x)$ we must have that $vH = vh - v\hat{h}$. However, this is not the case for $H = F - G_j$ as in the above example. This gives us the following corollary.

Corollary 5.1.4. *The ultrametric u on $K(x)$ does not come from a valuation.*

5.2 Root continuity for rational function fields

We will now look at the topic of continuity of roots with respect to u. For $F = \frac{f}{\hat{f}} \in K(x)$, an element $a \in K$ is called *a root of* F if it is a root of f, and *a pole of* F if it is a root of $\hat{f}$. Recall that we are assuming that f and $\hat{f}$ are coprime, hence an element cannot be simultaneously a root and a pole of F.

For $F, G \in K(x)$, we fix the following notation:

$$\begin{cases} F(x) := \frac{f(x)}{\hat{f}(x)} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{\hat{n}} \hat{a}_i x^i} = \frac{a_n \prod_{i=1}^{n}(x-\alpha_i)}{\hat{a}_{\hat{n}} \prod_{i=1}^{\hat{n}}(x-\hat{\alpha}_i)}, & n \geq 1, \hat{n} \geq 1, \\ G(x) := \frac{g(x)}{\hat{g}(x)} = \frac{\sum_{i=0}^{m} b_i x^i}{\sum_{i=0}^{m} \hat{b}_i x^i} = \frac{b_m \prod_{i=1}^{m}(x-\beta_i)}{\hat{b}_{\hat{m}} \prod_{i=1}^{\hat{m}}(x-\hat{\beta}_i)}, & m \geq 1, \hat{m} \geq 1. \end{cases} \tag{5.1}$$

Analogously to the case of polynomials, we define:

$$\deg\left(\frac{f}{\hat{f}}\right) := \max\{\deg f, \deg \hat{f}\},$$

$$\gamma(F) := \min\{\gamma(f), \gamma(\hat{f})\}, \quad \gamma^*(F) := \min\{\gamma(F), 0\}$$

and

$$\text{kras}(F) := \max\{\text{kras}(f), \text{kras}(\hat{f})\}.$$

Note that in this setting, the results we will prove for rational functions cannot be applied to polynomials, since (5.1) assumes that both the numerator and the denominator are not the constant polynomial.

Theorem 5.2.1. *Take* $\varepsilon \geq \max\{0, \text{kras}(F)\}$ *and* $F, G \in K(x)$ *as in (5.1) with* $\deg G \geq \deg F$*. Write*

$$d := \max\{\deg(f-g), \deg(\hat{f}-\hat{g})\}$$

and assume that

$$u(F,G) > \deg(F)\varepsilon - d\gamma^*(F) + \max\{va_n, 0\}. \tag{5.2}$$

Denote by t_k *and* $\hat{t}_k$ *the multiplicity of* α_k *and* $\hat{\alpha}_k$ *respectively. Then there exists an enumeration of roots and poles of* F *and* G *such that*

$$v(\alpha_k - \beta_k) > t_k\varepsilon \text{ for all } k \in \{1, \ldots, n\}$$

and

$$v(\hat{\alpha}_k - \hat{\beta}_k) > \hat{t}_k\varepsilon \text{ for all } k \in \{1, \ldots, \hat{n}\}.$$

Proof. Note that $\varepsilon \geq \text{kras}(\hat{f})$. Moreover,

$$\begin{aligned} v(\hat{f}-\hat{g}) &> \deg(F)\varepsilon - d\gamma^*(F) + \max\{va_n, 0\} \\ &\geq \deg(\hat{f})\varepsilon - \deg(\hat{f}-\hat{g})\gamma^*(\hat{f}). \end{aligned}$$

Since $\hat{f}$ and $\hat{g}$ are monic polynomials and $v(\hat{f}-\hat{g}) > 0$, we must have that $\deg \hat{f} = \deg \hat{g}$. This means that we can apply Theorem 4.2.4 to $\hat{f}$ and $\hat{g}$. We obtain that, after a suitable rearranging of indices, $v(\hat{\alpha}_k - \hat{\beta}_k) > \hat{t}_k$ for all $k \in \{1, \ldots, \hat{n}\}$.

We now wish to use an analogous argument for the polynomials f and g. Indeed, we have that $\varepsilon \geq \text{kras}(f)$ and

$$v(f-g) > \deg(f)\varepsilon - \deg(f-g)\gamma^*(f) + va_n.$$

We will now work with $\tilde{g} := a_n^{-1}g$ and the monic polynomial $\tilde{f} := a_n^{-1}f$. Our condition now reads:

$$v(\tilde{f}-\tilde{g}) > \deg(\tilde{f})\varepsilon - \deg(\tilde{f}-\tilde{g})\gamma^*(\tilde{f}).$$

Note that $\text{kras}(\tilde{f}) = \text{kras}(f)$, hence also $\varepsilon \geq \text{kras}(\tilde{f})$. Finally, observe that since $v(\tilde{f}-\tilde{g}) > 0$ and $\tilde{f}$ is a monic polynomial, the coefficient of $\tilde{g}$ next to x^n must have value 0. This shows that $\deg \tilde{g} \geq \deg \tilde{f}$. We may now apply Theorem 4.2.4 to $\tilde{f}$ and $\tilde{g}$ to obtain that, after a suitable rearranging of indices, $v(\alpha_k - \beta_k) > t_k$ for all $k \in \{1, \ldots, n\}$. Since $\tilde{f}$ has the same roots as f and $\tilde{g}$ has the same roots as g, this finishes the proof. □

In the bound for root continuity for polynomials, the term d was simply equal to $\deg(f-g)$ (see e.g. Theorem 4.2.2). As of now, we do not know whether one can replace the term d in the bound in Theorem 5.2.1 with $\deg(F-G)$. It would be possible if for all rational functions F and G we had that

$$\deg(F-G) \geq \max\{\deg(f-g), \deg(\hat{f}-\hat{g})\}.$$

However, this is in general not true. Indeed, take

$$F(x) = \frac{x^4+2x^2}{x+1}, \quad G(x) = \frac{x^4+2x^3}{x+3},$$

over any field K with $\operatorname{char} K \neq 2$. We see that $\deg(F-G) = 2$, and $\deg(f-g) = 3$. However, this example is not a counterexample to a version of Theorem 5.2.1 with $\deg(F-G)$ in place of d. Indeed, let $\alpha \in \widetilde{K}$ be such that $\alpha^2+2=0$. Then 0, α and $-\alpha$ are roots of x^4+2x^2. Since $v\alpha = \frac{v2}{2}$ and $v2 \geq 0$ with respect to any valuation on K, we have that

$$u(F,G) = v2 \leq \frac{3}{2}v2 = v(2\alpha) \leq \operatorname{kras}(F).$$

In particular, we cannot have that

$$u(F,G) > \deg(F)\varepsilon - \deg(F-G)\gamma^*(F) + \max\{va_n, 0\}.$$

The question remains open whether there exists an example of $F, G \in K(x)$ where the above inequality is satisfied, but the assertions of Theorem 5.2.1 do not hold.

Note that the bound for $u(F,G)$ depends on both F and G. In general, we cannot specify a bound for $u(F,G)$ that is independent from the degree of G. The argument for this is the same as the one for polynomials (see Example 4.2.6).

However, for continuity of poles, we can specify a bound that only depends on $\hat{f}$. Indeed, we have that if $u(F,G) > 0$, then $\deg \hat{f} = \deg \hat{g}$ and so $\deg(\hat{f}-\hat{g}) \leq \hat{n}-1$. Then the claim for the poles $\hat{\alpha}_k$ and $\hat{\beta}_k$ in Theorem 5.2.1 holds under the assumption that

$$v(\hat{f}-\hat{g}) > \hat{n}\varepsilon - (\hat{n}-1)\gamma^*(\hat{f}).$$

Corollary 5.2.2. *Take $\varepsilon \geq \max\{0, \operatorname{kras}(F)\}$ and $F, G \in K(x)$ as in (5.1) with $\deg G = \deg F = N$. Assume that*

$$u(F,G) > N\varepsilon - N\gamma^*(F) + \max\{va_n, 0\}.$$

Denote by t_k and $\hat{t}_k$ the multiplicity of α_k and $\hat{\alpha}_k$ respectively. Then there exists an enumeration of roots and poles of F and G such that

$$v(\alpha_k - \beta_k) > t_k\varepsilon \text{ for all } k \in \{1, \dots, n\}$$

and

$$v(\hat{\alpha}_k - \hat{\beta}_k) > \hat{t}_k\varepsilon \text{ for all } k \in \{1, \dots, \hat{n}\}.$$

Proof. Since $\deg G = \deg F$, we have that the degrees of f, $\hat{f}$, g and $\hat{g}$ are all bounded by $\deg F$. We thus have that

$$\max\{\deg(\hat{f} - \hat{g}), \deg(f - g)\} \leq N,$$

and so we can apply Theorem 5.2.1 to obtain our result. □

A natural question arises whether we can formulate an analogous result on continuity of roots and poles for the ultrametric w on $K(x)$, possibly by changing the bounds given in Theorem 5.2.1 and Corollary 5.2.2. Example 5.1.3 shows that not every sequence convergent with respect to w is convergent with respect to u. That example uses rational functions for which some of the polynomials are constant. However, in general it is not true that every counterexample must include a constant polynomial in a numerator or a denominator.

Example 5.2.3. Consider $\mathbb{Q}$ with the 2-adic valuation and set $\alpha := \frac{1}{2}$. We define

$$F(x) := \frac{f}{\hat{f}} := \frac{x^2 + 1}{x}, \quad G_j(x) := \frac{g_j}{\hat{g}_j} := \frac{\alpha^j x^2 + \alpha^j}{x^2 + \alpha^j x + 1}.$$

Then

$$v(f\hat{g}_j - \hat{f}g_j) = v(x^4 + 2x^2 + 1) = 0,$$

and $v(\hat{f}\hat{g}_j) = v\alpha^j = -j$. Therefore,

$$w(F, G_j) = v(f\hat{g}_j - \hat{f}g_j) - v(\hat{f}\hat{g}_j) = j.$$

Hence, G_j converges to F with respect to w, but not with respect to u.

Observe that this example is not a counterexample to continuity of roots and poles with respect to w since the roots of f and g_j coincide, while one root of $\hat{g}_j$ converges to 0 (this can be seen directly from looking at $\mathrm{NP}_{\hat{g}_j}$).

Chapter 6

Applications

In this chapter we will prove a number of results using the root continuity theorems presented earlier. Moreover, we are able to say more about the roots and the irreducible factors of polynomials which are sufficiently close to each other. As before, we consider a valued field (K, v) and the extension $(\widetilde{K}, v)$. All algebraic extensions of K will be equipped with the corresponding restriction of v. We let K^c be the completion of (K, v), equipped with the canonical extension of v. The first result in this chapter is an application of Theorem 2.1.3. It can also be found in [15, Theorem 32.19].

Theorem 6.0.1. *The completion of a Henselian field is again Henselian.*

Proof. Take a monic polynomial $f \in \mathcal{O}_{K^c}[x]$ and assume that fv has a simple root $\zeta \in (K^c)v = Kv$. We wish to show that f admits a root $\alpha \in \mathcal{O}_{K^c}$ such that $\alpha v = \zeta$. Extend the valuation v to the algebraic closure of K^c. Since ζ is a simple root of fv, there is a unique root α of f which under this extension satisfies $\alpha v = \zeta$. If we show that $\alpha \in K^c$, then the proof will be finished. To this end, fix any $\varepsilon > 0$. We wish to show the existence of an element $\beta \in K$ such that $v(\alpha - \beta) \geq \varepsilon$.

By the definition of K^c, for any $\delta > 0$ we can find $g \in K[x]$ such that $v(f - g) > \delta$. Since f has integral coefficients, it follows that $g \in \mathcal{O}_K[x]$. We employ part (c) of Theorem 2.1.3 to find that if $v(f - g)$ is large enough, then g has a root β such that $v(\alpha - \beta) \geq \varepsilon$. We have to show that $\beta \in K$. Note that $v(f - g) > 0$, thus we have that $gv = fv$, so that also gv admits ζ as a simple root. The field (K, v) is assumed to be Henselian, so there is a root $\beta_0 \in \mathcal{O}_K$ of g such that $\beta_0 v = \zeta$. Since $v(\alpha - \beta) \geq \varepsilon > 0$, we have that $\beta v = \alpha v = \zeta = \beta_0 v$. But ζ is a simple root of gv, so $\beta = \beta_0 \in K$. $\square$

6.1 Continuity of factorizations for Henselian fields

In this section, we will state a result from [15] (Theorem 6.1.3). It says that in a Henselian field, separable polynomials which are sufficiently close to each other, have factorizations which are "close to each other". That is, the number of irreducible factors is the same for both polynomials, and those factors can be paired in such a way that they are close to each other.

Once this theorem is proved, in the next section we will work on finding a version thereof which does not assume that the polynomials in question are separable.

For further results, we will employ the following technical lemma.

Lemma 6.1.1. *Take an arbitrary valued field* (L, v) *and elements*

$$c_1, \ldots, c_n, d_1, \ldots, d_n \in L$$

such that $vc_i \leq 0$ *for all* i*. Take* $\varepsilon \geq 0$ *and assume that for* $1 \leq j \leq n$*,*

$$v(c_j - d_j) > \varepsilon - v \prod_{1 \leq i \leq n} c_i.$$

Then for every subset $I \subseteq \{1, \ldots, n\}$*,*

$$v\left(\prod_{i \in I} c_i - \prod_{i \in I} d_i\right) > \varepsilon. \tag{6.1}$$

Proof. Observe that since $vc_i \leq 0$ for all i, the value of any product of the c_i also does not exceed 0. Since $\varepsilon \geq 0$, it follows that $v(c_j - d_j) > 0$ for all j, which in turn implies that $vc_j = vd_j$.

By induction we show that for $1 \leq k \leq n$,

$$v\left(\prod_{1 \leq i \leq k} c_i - \prod_{1 \leq i \leq k} d_i\right) > \varepsilon - v \prod_{k+1 \leq i \leq n} c_i > \varepsilon, \tag{6.2}$$

where for $k = n$ we have that $v \prod_{i=n+1}^{n} c_i = v1 = 0$.

Given $I \subseteq \{1, \ldots, n\}$, we can without loss of generality renumber the elements c_i so that $I = \{1, \ldots, k\}$ for some k. Then (6.2) will prove (6.1).

Observe that (6.2) holds for $k = 1$ because

$$v(c_1 - d_1) > \varepsilon - v \prod_{1 \leq i \leq n} c_i \geq \varepsilon - v \prod_{2 \leq i \leq n} c_i.$$

Now assume that $1 \leq k < n$ and that (6.2) holds for k. We compute:

$$
\begin{aligned}
v\left(\prod_{1\leq i\leq k+1} c_i - \prod_{1\leq i\leq k+1} d_i\right) = \\
&= v\left(c_{k+1}\prod_{1\leq i\leq k} c_i - d_{k+1}\prod_{1\leq i\leq k} d_i\right)\\
&= v\left(c_{k+1}\prod_{1\leq i\leq k} c_i - d_{k+1}\prod_{1\leq i\leq k} c_i + d_{k+1}\prod_{1\leq i\leq k} c_i - d_{k+1}\prod_{1\leq i\leq k} d_i\right)\\
&\geq \min\left\{v(c_{k+1}-d_{k+1}) + v\prod_{1\leq i\leq k} c_i,\, vd_{k+1} + v\left(\prod_{1\leq i\leq k} c_i - \prod_{1\leq i\leq k} d_i\right)\right\}.
\end{aligned}
$$

By the assumption of our lemma,

$$
v(c_{k+1}-d_{k+1}) + v\prod_{1\leq i\leq k} c_i > \varepsilon - v\prod_{1\leq i\leq n} c_i + v\prod_{1\leq i\leq k} c_i \geq \varepsilon - v\prod_{k+2\leq i\leq n} c_i.
$$

By our induction assumption,

$$
\begin{aligned}
vd_{k+1} + v\left(\prod_{1\leq i\leq k} c_i - \prod_{1\leq i\leq k} d_i\right) &> vd_{k+1} + \varepsilon - v\prod_{k+1\leq i\leq n} c_i\\
&= vc_{k+1} + \varepsilon - v\prod_{k+1\leq i\leq n} c_i\\
&= \varepsilon - v\prod_{k+2\leq i\leq n} c_i.
\end{aligned}
$$

This shows that (6.2) holds for $k+1$ in place of k and completes the proof of our lemma. □

In the special case where (K,v) is a valued field and the rational function field $K(x)$ is endowed with the Gauß valuation, we can take $c_i = x - \alpha_i$ and $d_i = x - \beta_i$. Then $vc_i \leq 0$, $v(c_j - d_j) = v(\alpha_j - \beta_j)$ and $v\prod_{i=1}^n c_i = v\prod_{i=1}^n (x-\alpha_i)$. Thus with $L = K(x)$, the above lemma yields:

Corollary 6.1.2. *Take a valued field (K,v) and $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n \in K$. Choose a non-negative value $\varepsilon \in vK$ and assume that for $1 \leq j \leq n$,*

$$
v(\alpha_j - \beta_j) > \varepsilon - v\prod_{1\leq i\leq n}(x-\alpha_i).
$$

Then for every subset $I \subseteq \{1, \ldots, n\}$,

$$v\left(\prod_{i\in I}(x-\alpha_i) - \prod_{i\in I}(x-\beta_i)\right) > \varepsilon. \tag{6.3}$$

The following is Theorem 32.20 from [15]. It allows us to obtain information on the irreducible factors of polynomials which are sufficiently close to each other. We give a proof by use of Theorem 2.1.3.

Theorem 6.1.3. *Let (K, v) be a Henselian field and $f = f_1 \cdot \ldots \cdot f_r$ where $f_1, \ldots, f_r$ are distinct monic separable irreducible polynomials over K. Then for every $\varepsilon > \max\{0, \mathrm{kras}(f)\}$ there is some $\delta \in vK$ such that for every monic polynomial $g \in K[x]$ satisfying $v(f-g) > \delta$ we have that $g = g_1 \cdot \ldots \cdot g_r$, where $g_1, \ldots, g_r$ are distinct monic separable irreducible polynomials over K. Moreover, for each $k \in \{1, \ldots, r\}$ the following assertions hold:*

(a) $\deg f_k = \deg g_k$ *and* $v(f_k - g_k) > \varepsilon$*,*

(b) for every root α of f_k there exists a root β of g_k such that $K(\alpha) = K(\beta)$,

(c) f_k and g_k have the same splitting field,

(d) for all roots α of f_k and β of g_k, $K(\alpha)$ and $K(\beta)$ are isomorphic over K.

Proof. Let $n = \deg f$ and choose any $\varepsilon > \max\{0, \mathrm{kras}(f)\}$. By assumption, f has n distinct roots $\alpha_1, \ldots, \alpha_n \in \widetilde{K}$. We take any δ satisfying

$$\delta \geq n\left(\varepsilon - vf - \gamma^*(f)\right).$$

Then the assumption $v(f-g) > \delta$ implies that

$$\varepsilon_0 := \frac{v(f-g)}{n} + \gamma^*(f) > \varepsilon - vf \geq \varepsilon > \max\{0, \mathrm{kras}(f)\}.$$

By Theorem 2.1.3, for every α_i there exists a unique root β_i of g satisfying $v(\alpha_i - \beta_i) > \varepsilon_0$. Consequently, g is separable.

For every $k \in \{1, \ldots, r\}$, we define $g_k = \prod(x - \beta_i)$, where the product is taken over all i such that α_i is a root of f_k. Then the factors g_k are separable and pairwise distinct, and $\deg f_k = \deg g_k$. Thus it suffices to show that each g_k is an irreducible polynomial over K. Let α_i be a root of f_k; then by construction, β_i is a root of g_k. Assume that $\sigma\alpha_i = \alpha_j$. Since (K, v) is Henselian, we have that $v(\alpha_j - \sigma\beta_i) = v\sigma(\alpha_i - \beta_i) = v(\alpha_i - \beta_i) > \varepsilon_0$. As β_j is the unique root of g such that $v(\alpha_j - \beta_j) > \varepsilon_0$, it follows that $\sigma\beta_i = \beta_j$. Therefore, every $\sigma \in \mathrm{Gal}\, K$ maps the roots of g_k onto roots of g_k, and thus

g_k is a polynomial over K. Conversely, let β_i and β_j be two roots of g_k. Since f_k is irreducible over K, we can find $\sigma \in \operatorname{Gal} K$ such that $\sigma\alpha_i = \alpha_j$. By the same argument as before we find that $\sigma\beta_i = \beta_j$, which means that g_k must be irreducible.

Since for each $1 \leq i \leq n$ we have that $v(\alpha_i - \beta_i) > \varepsilon_0 > \varepsilon - vf$, we can employ Corollary 6.1.2 for the elements α_i and β_i to obtain:

$$\forall_{1\leq k\leq r}\ v(f_k - g_k) > \varepsilon.$$

This proves assertion (a).

Fix any root β_i of g. Since g is separable over K, β_i lies in K^{sep}. Assume first that $\beta_i \in K$. Then the corresponding irreducible polynomial is of the form $x - \beta_i$. Thus $K(\beta_i) = K = K(\alpha_i)$, in which case assertion (b) of the theorem holds. Now assume that $\beta_i \in K^{\text{sep}} \setminus K$. By our choice of ε_0 and by Krasner's Lemma (Lemma 4.2.1) we obtain that $K(\alpha_i) \subseteq K(\beta_i)$. But if k is such that α_i is a root of f_k, then $[K(\alpha_i) : K] = \deg f_k = \deg g_k = [K(\beta_i) : K]$, showing that $K(\alpha_i) = K(\beta_i)$. This proves assertion (b), which readily implies assertions (c) and (d). □

From the above theorem we derive the following result.

Corollary 6.1.4. *Let (K, v) be a an arbitrary valued field and take a monic separable polynomial $f \in K[x]$. Assume that f has a factorization into distinct irreducible polynomials over K^h of the form $f = f_1 \cdot \ldots \cdot f_r$. Then for every $\varepsilon > \max\{0, \operatorname{kras}(f)\}$ there is some $\delta \in vK$ such that for every monic polynomial $g \in K[x]$ satisfying $v(f - g) > \delta$ we have that $g = g_1 \cdot \ldots \cdot g_r$, where $g_1, \ldots, g_r$ are distinct monic separable irreducible polynomials over K^h. Moreover, for each $k \in \{1, \ldots, r\}$, assertions (a)–(d) of Theorem 6.1.3 hold for K^h in place of K.*

6.2 Irreducible factors over the henselization and double cosets in the Galois group

In this section, we will have a closer look at the polynomials in a factorization of a given polynomial over a henselization of a field K. We will then study the connection between the irreducible factors and representatives of *double cosets*. To this end, we will employ notions from [9, Section 7.9]. Let H_1, H_2 be subgroups of a group G. Then for $g \in G$ the set

$$H_1 g H_2 = \{h_1 g h_2 \mid h_1 \in H_1 \wedge h_2 \in H_2\}$$

is called a *double coset of* G. The set of all such double cosets induces an equivalence relation on G, with respect to which two elements g_1, g_2 are equivalent if $H_1 g_1 H_2 = H_1 g_2 H_2$. For each element in the corresponding equivalence class we may then choose a representative. This notion will be employed for subgroups of the group $\operatorname{Gal} K$.

Notation 6.2.1. Throughout, we will consider a finite extension $L|K$ and fix the representatives $\iota_1, \ldots, \iota_s \in \operatorname{Gal} K$ of the double cosets

$$\{(\operatorname{Gal} K^h)\iota(\operatorname{Gal} L) \mid \iota \in \operatorname{Gal} K\}.$$

Note that $s \leq (\operatorname{Gal} K : \operatorname{Gal} L) \leq [L : K] < \infty$. For $\iota \in \operatorname{Gal} K$ we denote by $\operatorname{res}_L(\iota) = \iota|_L$ the restriction of ι to L. Further, by $[L : K]_{\text{sep}}$ we denote the degree of the maximal separable subextension of $L|K$, and we set

$$[L : K]_{\text{ins}} := \frac{[L : K]}{[L : K]_{\text{sep}}}.$$

We define the *characteristic exponent of* K to be $\operatorname{charexp} K := \operatorname{char} K$ if $\operatorname{char} K > 0$, and $\operatorname{charexp} K := 1$ otherwise. Then $[L : K]_{\text{ins}}$ is a power of $\operatorname{charexp} K$ for any finite extension $L|K$.

The following two lemmas can be found in [9, Lemma 7.46] in a more general form, with an arbitrary algebraic extension K' in place of K^h. For our purposes the result in the simplified form is sufficient.

Lemma 6.2.2. *An automorphism* $\iota \in \operatorname{Gal} K$ *lies in* $\operatorname{Gal} K^h \iota_i \operatorname{Gal} L$ *if and only if the isomorphism* $\operatorname{res}_{\iota_i L}(\iota\iota_i^{-1}) : \iota_i L \to \iota L$ *can be extended to an isomorphism of* $(\iota_i L).K^h$ *onto* $\iota L.K^h$ *over* K^h*.*

Proof. Take $\iota \in \operatorname{Gal} K$. Then an automorphism in $\operatorname{Gal} K$ extends $\operatorname{res}_{\iota_i L}(\iota\iota_i^{-1})$ if and only if it lies in the coset $\iota\iota_i^{-1} \operatorname{Gal} \iota_i L$. This coset is equal to

$$\iota\iota_i^{-1}\iota_i(\operatorname{Gal} L)\iota_i^{-1} = \iota(\operatorname{Gal} L)\iota_i^{-1}.$$

Hence, there is an extension of $\operatorname{res}_{\iota_i L}(\iota\iota_i^{-1})$ to an isomorphism over K^h if and only if

$$\iota(\operatorname{Gal} L)\iota_i^{-1} \cap \operatorname{Gal} K^h \neq \emptyset.$$

But this is equivalent to $\iota \in (\operatorname{Gal} K^h)\iota_i \operatorname{Gal} L$. □

Lemma 6.2.3. *Consider* $L|K$ *as in Notation 6.2.1 and let* K_s *be the maximal separable subextension of* $L|K$*. Assume that* $K_s = K(\alpha)$ *and take* f *to be the minimal polynomial of* α *over* K*. Let* $f = f_1 \cdot \ldots \cdot f_r$ *be the factorization* *of f into irreducible polynomials over K^h. Then $r = s$, and after suitably rearranging indices we have that $\iota_i\alpha$ is a root of f_i, so that*

$$[(\iota_i K_s).K^h : K^h] = \deg f_i.$$

Moreover, the following equalities hold:

$$[L : K]_{\text{ins}} = [(\iota_i L).K^h : K^h]_{\text{ins}}, \quad 1 \leq i \leq s, \tag{6.4}$$

$$[L : K] = \sum_{1 \leq i \leq s} [(\iota_i L).K^h : K^h]. \tag{6.5}$$

Proof. Observe that since $K_s|K$ is finite and separable, we can always find α such that $K_s = K(\alpha)$.

We will first prove Equation (6.4). Note that $L|K_s$ is purely inseparable and thus, $\operatorname{Gal} L = \operatorname{Gal} K_s$. As $K^h|K$ is separable, so is $(\iota_i K_s).K^h|\iota_i K_s$. Since $\iota_i L|\iota_i K_s$ is purely inseparable, it is linearly disjoint from $(\iota_i K_s).K^h|\iota_i K_s$, and $(\iota_i L).K^h|(\iota_i K_s).K^h$ is purely inseparable. This yields that $[\iota_i L : \iota_i K_s] = [(\iota_i L).K^h : (\iota_i K_s).K^h]$ and that $(\iota_i K_s).K^h|K^h$ is the maximal separable subextension of $(\iota_i L).K^h|K^h$. Hence,

$$[L : K]_{\text{ins}} = [\iota_i L : \iota_i K_s] = [(\iota_i L).K^h : (\iota_i K_s).K^h] = [(\iota_i L).K^h : K^h]_{\text{ins}}.$$

Consider α and $f = f_1 \cdot \ldots f_r$ as in the assumption of the theorem. Then $\operatorname{res}_{\iota_i K_s}(\iota\iota_i^{-1}) : \iota_i K_s \to \iota K_s$ can be extended to an isomorphism of $(\iota_i K_s).K^h$ onto $(\iota K_s).K^h$ over K^h if and only if $\iota_i\alpha$ and $\iota\alpha$ are roots of the same irreducible factor. We apply Lemma 6.2.2 to K_s in place of L to obtain that there are s many such factors, and we may enumerate them such that $\iota_i\alpha$ is a root of f_i. Then $[(\iota_i K_s).K^h : K^h] = \deg f_i$. Hence,

$$[L : K]_{\text{sep}} = [K_s : K] = \deg f = \sum_{1 \leq i \leq s} \deg f_i = \sum_{1 \leq i \leq s} [(\iota_i K_s).K^h : K^h]. \tag{6.6}$$

Since the extension $(\iota_i L).K^h|(\iota_i K_s).K^h$ is purely inseparable, we have that

$$[(\iota_i K_s).K^h : K^h] = [(\iota_i L).K^h : K^h]_{\text{sep}} = [(\iota_i L).K^h : K^h] \cdot [(\iota_i L).K^h : K^h]_{\text{ins}}^{-1}.$$

In view of this equality and by Equation (6.4), multiplying Equation (6.6) with $[L : K]_{\text{ins}}$ yields Equation (6.5). □

Assume that $\operatorname{charexp} K = p$. For $f \in K[x]$ denote by $\operatorname{ins} f$ the *degree of inseparability of* f, that is, the maximal number p^ν which divides every exponent in $f(x)$. In this case $f(x)$ can be written as $\tilde{f}(x^{p^\nu})$ for some $\tilde{f} \in K[x]$, and $\operatorname{ins} \tilde{f} = 1$.

Lemma 6.2.4. *Fix any irreducible polynomial $f \in K[x]$ and let α be a root of f. Assume that $\operatorname{ins} f = p^\nu$ and take $\tilde{f} \in K[x]$ such that $\tilde{f}(x^{p^\nu}) = f(x)$. Let $\tilde{f} = \tilde{f}_1 \cdot \ldots \cdot \tilde{f}_r$ be the factorization of $\tilde{f}$ into irreducible polynomials over K^h and set $f_i(x) := \tilde{f}_i(x^{p^\nu})$. Then $f = f_1 \cdot \ldots \cdot f_r$ is the factorization of f into irreducible polynomials over K^h. Moreover, for $\iota_1, \ldots, \iota_s$ chosen as in Notation 6.2.1 for $L = K(\alpha)$, and after suitably rearranging indices, $\iota_i \alpha$ is a root of f_i and $\deg f_i = [(\iota_i K(\alpha)).K^h : K^h]$. In particular, $r = s$.*

Proof. We observe that $\tilde{f}$ is irreducible over K since every factorization $\tilde{f} = \tilde{g}\tilde{h}$ leads to a factorization $f = \tilde{g}(x^{p^\nu})\tilde{h}(x^{p^\nu})$. Moreover, $\tilde{f}$ is separable. Indeed, if it were inseparable, then by its irreducibility we would have that $\tilde{f}' \equiv 0$.

But this would mean that every exponent in $\tilde{f}$ is divisible by p, which contradicts the construction of $\tilde{f}$.

We have that α is a root of f if and only if α^{p^ν} is a root of $\tilde{f}$. Thus the extension $K(\alpha^{p^\nu})|K$ is separable, and $K(\alpha)|K(\alpha^{p^\nu})$ is purely inseparable. Therefore, if K_s is the maximal separable subextension of K in $K(\alpha)$, then $K_s = K(\alpha^{p^\nu})$. We apply Lemma 6.2.3 for $\tilde{f}$ in place of f and α^{p^ν} in place of α. We obtain that $\tilde{f}$ splits into irreducible factors $\tilde{f}_1, \ldots, \tilde{f}_s$ over K^h such that $(\iota_i\alpha)^{p^\nu}$ is a root of $\tilde{f}_i$, and

$$\deg \tilde{f}_i = [(\iota_i K_s).K^h : K^h].$$

In particular, $\tilde{f} = \tilde{f}_1 \cdot \ldots \cdot \tilde{f}_s$ is precisely the factorization of $\tilde{f}$ into irreducible polynomials over K^h, hence $r = s$. We define $f_i(x) := \tilde{f}_i(x^{p^\nu})$, then $f = f_1 \cdot \ldots \cdot f_s$ and $\iota_i\alpha$ is a root of f_i. We observe that $\iota_i K(\alpha) = K(\iota_i\alpha)$ and $\iota_i K_s = K\left((\iota_i\alpha)^{p^\nu}\right)$. Since $K^h|K$ is separable, also $\left(K\left((\iota_i\alpha)^{p^\nu}\right).K^h\right)|K\left((\iota_i\alpha)^{p^\nu}\right)$ is separable and thus linearly disjoint from $K(\iota_i\alpha)|K\left((\iota_i\alpha)^{p^\nu}\right)$. Therefore,

$$\begin{aligned}
[K(\iota_i\alpha).K^h : K^h] &= [(\iota_i K(\alpha)).K^h : K^h], \\
[K((\iota_i\alpha)^{p^\nu}).K^h : K^h] &= [(\iota_i K_s).K^h : K^h], \\
[K(\iota_i\alpha).K^h : K((\iota_i\alpha)^{p^\nu}).K^h] &= [K(\iota_i\alpha) : K((\iota_i\alpha)^{p^\nu})].
\end{aligned}$$

Consequently, the equality

$$[K(\iota_i\alpha).K^h : K^h] = [K(\iota_i\alpha).K^h : K((\iota_i\alpha)^{p^\nu}).K^h] \cdot [K((\iota_i\alpha)^{p^\nu}).K^h : K^h]$$

implies that

$$[(\iota_i K(\alpha)).K^h : K^h] = p^\nu [(\iota_i K_s).K^h : K^h] = p^\nu \deg \tilde{f}_i = \deg f_i.$$

This shows that the f_i are irreducible over K^h. □

The assumption on the separability of f in Corollary 6.1.4 can be dropped at the cost of adding an assumption on $\operatorname{ins} f$ and $\operatorname{ins} g$.

Theorem 6.2.5. *Let (K, v) be an arbitrary field and take $f \in K[x]$ monic and irreducible over K. Assume that f has a factorization into distinct irreducible polynomials over K^h of the form $f = f_1 \cdot \ldots \cdot f_r$. For $\operatorname{ins} f = p^\nu$ take $\tilde{f} \in K[x]$ such that $\tilde{f}(x^{p^\nu}) = f(x)$. Then for every $\varepsilon > \max\{0, \operatorname{kras}(\tilde{f})\}$ there is some $\delta \in vK$ such that the following holds: If g is any irreducible monic polynomial over K satisfying $\operatorname{ins} g \geq \operatorname{ins} f$ and $v(f - g) > \delta$, then:*

- $\deg f = \deg g$ *and* $\operatorname{ins} g = \operatorname{ins} f$,
- $g = g_1 \cdot \ldots \cdot g_r$ *where $g_1, \ldots, g_r$ are irreducible polynomials over K^h,*
- *for each $k \in \{1, \ldots, r\}$, assertions (a)–(d) of Theorem 6.1.3 hold with K^h in place of K.*

Proof. For f as given in (2.1), we take $\delta \geq \max\{va_i \mid 1 \leq i \leq n \wedge a_i \neq 0\}$. Choose any irreducible monic polynomial g such that $v(f-g) > \delta$. Then $a_i \neq 0$ implies that $b_i \neq 0$, so then $\deg f = \deg g$. Moreover, since $\operatorname{ins} f$ divides every i such that $a_i \neq 0$, we must also have that $\operatorname{ins} g \leq \operatorname{ins} f$. Together with the hypothesis that $\operatorname{ins} g \geq \operatorname{ins} f$ we then obtain that $\operatorname{ins} g = \operatorname{ins} f$; let us assume that it is equal to p^ν.

Let $\tilde{f}$, $\tilde{g} \in K[x]$ be such that $\tilde{f}(x^{p^\nu}) = f(x)$, $\tilde{g}(x^{p^\nu}) = g(x)$. Then $\tilde{f}$ is separable and irreducible over K. Let $\tilde{f} = \tilde{f}_1 \cdot \ldots \cdot \tilde{f}_r$ be the factorization of $\tilde{f}$ into irreducible polynomials over K^h. By Lemma 6.2.4, $\tilde{f}_i(x^{p^\nu}) = f_i(x)$; in particular, the polynomials $\tilde{f}_i$ are distinct. Since $v(\tilde{f} - \tilde{g}) = v(\tilde{f}(x^{p^\nu}) - \tilde{g}(x^{p^\nu})) = v(f - g)$, we may apply Corollary 6.1.4 to $\tilde{f}$ and $\tilde{g}$ in the place of f and g, enlarging the originally chosen δ if necessary (note that this value only depends on f, not on g). We obtain that the respective factorization of $\tilde{g}$ into distinct irreducible polynomials over K^h is of the form $\tilde{g} = \tilde{g}_1 \cdot \ldots \cdot \tilde{g}_r$.

Set $g_k(x) := \tilde{g}_k(x^{p^\nu}) \in K^h[x]$. Since g is irreducible and $\operatorname{ins} g = \operatorname{ins} f = p^\nu$, by Lemma 6.2.4 we see that $g = g_1 \cdot \ldots \cdot g_r$ is precisely the factorization of g into irreducible polynomials over K^h.

Since a root α^{p^ν} of $\tilde{f}_k$ corresponds to a root α of f_k, the last assertion follows from Corollary 6.1.4 applied again to $\tilde{f}$ and $\tilde{g}$. $\square$

Remark 6.2.6. Assume that for the monic irreducible polynomial f as in (2.1) and the corresponding $\delta \geq \max\{va_i \mid 1 \leq i \leq n \wedge a_i \neq 0\}$ we have already chosen a monic irreducible polynomial g as in (2.1) such that $v(f - g) > \delta$. We leave it to the reader to observe that, under these assumptions, the following properties are equivalent:

(a) we can find $\delta' \geq \delta$ such that $v(f-g) > \delta'$ and

$$\delta' \geq \max(\{va_i \mid 1 \leq i \leq n \wedge a_i \neq 0\} \cup \{vb_i \mid 1 \leq i \leq n \wedge b_i \neq 0\}),$$

(b) for all $i \in \{1, \ldots, n\}$ we have that $va_i = vb_i$,

(c) for all $i \in \{1, \ldots, n\}$ we have that $b_i \neq 0 \Rightarrow a_i \neq 0$.

If the above conditions are satisfied, then in the same way as in the preceding proof we can show that $\operatorname{ins} g = \operatorname{ins} f$. In this case the hypothesis "$\operatorname{ins} g \geq \operatorname{ins} f$" is therefore not needed. Note that conditions (a)–(c) are stronger than this hypothesis.

Example 6.2.7. We claim that there exist monic irreducible polynomials f and g such that $v(f-g) > \max\{va_i \mid 1 \leq i \leq n \wedge a_i \neq 0\}$, but $\operatorname{ins} g < \operatorname{ins} f$. To this end, we will employ a construction which can be found in [10].

Consider the field $K := \mathbb{F}_p(t)$ with the t-adic valuation and take the element $c := \frac{1}{t}$. Then the monic polynomial $f(x) := x^p - c$ is irreducible over K. Take $d \in K$ such that $vd > 0$, define $g(x) := x^p - d^{p-1}x - c$ and take a root θ of g. Then $\frac{\theta}{d}$ is a root of the Artin–Schreier polynomial $x^p - x - \frac{c}{d^p}$. Since $v\frac{c}{d^p} < 0$, we have that $p \cdot v\frac{\theta}{d} = v\frac{c}{d^p}$ ([10, Lemma 2.27]), and so $v\theta = \frac{vc}{p}$. This means that $\theta \notin K$. Since all the roots of g are of the form $\theta + d \cdot i$, where $i \in \mathbb{F}_p$, this shows that the polynomial g is irreducible over K. The polynomials f and g thus satisfy our claim.

The irreducibility of g in Theorem 6.2.5 is essential for assuring that the corresponding factorization of g over K^h yields irreducible polynomials:

Example 6.2.8. We claim that there exists $f \in K[x]$ monic and irreducible over K such that for every $\delta \in vK$ there exists a monic polynomial $g \in K[x]$ satisfying $\operatorname{ins} g = \operatorname{ins} f$ and $v(f-g) > \delta$, but f and g do not split into the same number of irreducible factors over K^h, and assertions (a)–(d) of Theorem 6.1.3 do not hold for any choice of $\varepsilon > 0$ and any of the polynomials f_k, g_k, and their respective roots.

Take (k, v) to be the rational function field $\mathbb{F}_p(t)$ with the t-adic valuation, extended canonically to $\mathbb{F}_p((t))$. Since the transcendence degree of $\mathbb{F}_p((t))$ over $\mathbb{F}_p(t)$ is infinite, we can choose an element in $\mathbb{F}_p((t))$ transcendental over k. For example, take $z := \sum_{i=1}^{\infty} t^{p \cdot i!} \in \mathbb{F}_p((t))$ and define $K := k(z)$. Consider the purely inseparable extension $k(z^{\frac{1}{p}})|K$ of degree p. Observe that $z^{\frac{1}{p}} = \sum_{i=1}^{\infty} t^{i!} \in \mathbb{F}_p((t))$, thus $z^{\frac{1}{p}} \in K^c$. Take f to be the minimal polynomial of $z^{\frac{1}{p}}$ over K, that is, $f(x) = x^p - z$.

To prove our claim, fix any element $\delta \in vK$ and assume without loss of generality that $\delta > 0$. Since $z^{\frac{1}{p}} \in K^c$, we can find an element $\beta \in K$ such that $v(z^{\frac{1}{p}} - \beta) > \delta$. (In fact, we can take $\beta = \sum_{i=1}^{n} t^{i!}$ for n large enough.) Consider the polynomial $g(x) = x^p - \beta^p \in K[x]$, then $\operatorname{ins} f = \operatorname{ins} g$ and $v(f - g) = v(z - \beta^p) > p\delta > \delta$. Since $K^h|K$ is separable and $f(x)$ is purely inseparable, we cannot have $z^{\frac{1}{p}} \in K^h$, so $f(x)$ must be irreducible over K^h. On the other hand, g splits into p linear factors already over K, so in particular over K^h. Clearly, $K^h(z^{\frac{1}{p}})$ cannot be equal nor isomorphic to $K^h(\beta) = K^h$ over K^h.

6.3 The Fundamental Equality and ramification theoretical applications of root continuity

In this section, we will have a closer look at properties of henselizations. We will work with Notation 6.2.1 in order to study the connection between the representatives of the double cosets and the henselizations with respect to different extensions of the valuation v. This will allow us to give a proof of the well-known *Fundamental Equality*. Moreover, we will show that if two polynomials are close to each other, then their roots give rise to extensions which have the same ramification theoretical invariants.

For the next results we will require a number of properties of the henselization K^h. The extension $(K^h|K, v)$ is *immediate* ([5, Corollary 5.3.8]). Note that any algebraic extension of K^h is again a Henselian field. Thus if $(K, v) \subset (E, v) \subset (\widetilde{K}, v)$, then we have that

$$(E^h, v) = (E.K^h, v). \tag{6.7}$$

Take any $\sigma \in \operatorname{Gal}(\widetilde{K}|K)$. Then the map

$$v\sigma = v \circ \sigma : E \ni a \mapsto v(\sigma a) \in v\widetilde{K}$$

is a valuation on E which extends K. In fact, all extensions of v from K to E are *conjugate*. That is, all the extensions are of the form $v\sigma$, where σ is an embedding of E in $\widetilde{K}$ over K ([5, Theorem 3.2.15]).

Consider the group

$$G^d := G^d(\widetilde{K}|K, v) := \{\sigma \in \operatorname{Gal} K \mid v(\sigma x) = vx \text{ for all } x \in \widetilde{K}\}$$

called the *decomposition group of* $(\widetilde{K}|K, v)$, and its *fixed field*

$$K^d := \operatorname{Fix}(G^d) := \{a \in \widetilde{K} \mid \sigma(a) = a \text{ for all } \sigma \in G^d\}.$$

We will also write $(\widetilde{K}|K,v)^d$ in place of K^d to specify which valuation we are considering. This field is the henselization of K with respect to the valuation v. For more details on ramification theory, see e.g. [4, Chapter 3], [5, Sect. 5.2], [9, Chapter 7] or [11, Sect. 2.9].

For the convenience of the reader we include a number of results on G^d and K^d. The following statements can be found in [9, Sect. 7].

Lemma 6.3.1. *Take $\iota, \sigma, \tau \in \operatorname{Gal} K$.*

(a) We have that $(\widetilde{K}|K, v\iota)^d = \iota^{-1}K^d$.

(b) If $v\sigma = v\tau$ on $\tau^{-1}K^d$, then $v\sigma = v\tau$ on $\widetilde{K}$ and $\sigma\tau^{-1} \in G^d$.

(c) The restriction $\operatorname{res}_{K^h}(\iota^{-1})$ is the unique isomorphism over K sending K^d onto $(\widetilde{K}|K, v\iota)^d$.

Proof. Observe that $G^d(\widetilde{K}|K, v\iota) = \iota^{-1}\left(G^d(\widetilde{K}|K,v)\right)\iota$ ([11, Proposition 9.4], [4, (15.2)]). Assertion (a) thus follows since $\operatorname{Fix}(\iota^{-1}G\iota) = \iota^{-1}\operatorname{Fix}(G)$ for any automorphism group G.

Consider now σ and τ as in (b), then $v\sigma\tau^{-1} = v$ on K^d. Since the extension of v from K^d to $\widetilde{K}$ is unique, $v\sigma\tau^{-1} = v$ also holds on $\widetilde{K}$. Assertion (b) then follows from the definition of G^d.

It follows from part (a) that the restriction of ι^{-1} is the required isomorphism. If there were a second isomorphism, say σ^{-1}, then $v\sigma = v\iota$ on $\iota^{-1}K^d$, so by part (b), ι^{-1} and σ^{-1} must coincide on K^d. This proves part (c). □

If w is another extension of v from K to $\widetilde{K}$, then we will denote by $K^{h(w)}$ the henselization of (K,v) in $(\widetilde{K}, w)$ The above lemma allows us to represent extensions of v from K to K^h by means of the automorphism ι.

Lemma 6.3.2. *For every $\iota \in \operatorname{Gal} K$, the field $(\iota^{-1}K^h, v\iota)$ is the henselization $(K^{h(v\iota)}, v\iota)$ of (K,v) in $(\widetilde{K}, v\iota)$, and (K^h, v) is isomorphic over K to $(\iota^{-1}K^h, v\iota)$ via the uniquely determined isomorphism $\operatorname{res}_{K^h}(\iota^{-1})$.*

Proof. The assertion follows from the definition of the henselization together with part (a) of Lemma 6.3.1. The uniqueness of $\operatorname{res}_{K^h}(\iota^{-1})$ comes from part (c) of that lemma. □

Lemma 6.3.3. *Let $\iota_1, \ldots, \iota_s$ and L be as in Notation 6.2.1, and write $v_i := v\iota_i$. Then $(L.\iota_i^{-1}K^h, v_i)$ is the henselization of (L, v_i) in $(\widetilde{K}, v_i)$, and it is isomorphic over K to $(\iota_i L.K^h, v)$ via ι_i. Further, the distinct extensions of v from K to L are precisely the restrictions of the valuations v_i to L, $1 \le i \le s$.*

Proof. By virtue of Lemma 6.3.2, $(\iota_i^{-1}K^h, v_i)$ is the henselization of (K, v) in $(\widetilde{K}, v_i)$. From Equation (6.7) it follows that $(L.\iota_i^{-1}K^h, v_i)$ is the henselization of (L, v_i) in $(\widetilde{K}, v_i)$. The restriction of ι_i is an isomorphism from $(L.\iota_i^{-1}K^h, v\iota_i)$ onto $(\iota_i L.K^h, v)$ over K.

Assume that for some $\iota \in \operatorname{Gal} K$ we have that $v\iota = v\iota_i$ on L. Then $v\iota$ and $v\iota_i$ are both extensions of the same valuation from L to $\widetilde{K}$. Since those extensions are conjugate, there exists $\tau \in \operatorname{Gal} L$ such that $v\iota_i\tau = v\iota$ on $\widetilde{K}$. By Lemma 6.3.2, $\iota^{-1}K^h = \tau^{-1}\iota_i^{-1}K^h$ is the henselization of K in $(\widetilde{K}, v\iota) = (\widetilde{K}, v\iota_i\tau)$, so the restrictions of ι^{-1} and $\tau^{-1}\iota_i^{-1}$ to K^h must be equal. Hence, $\sigma := \iota\tau^{-1}\iota_i^{-1} \in \operatorname{Gal} K^h$ and thus $\iota = \sigma\iota_i\tau \in (\operatorname{Gal} K^h)\iota_i(\operatorname{Gal} L)$.

For the converse, assume that $\iota \in (\operatorname{Gal} K^h)\iota_i(\operatorname{Gal} L)$. Write $\iota = \sigma\iota_i\tau$ with $\sigma \in \operatorname{Gal} K^h$ and $\tau \in \operatorname{Gal} L$. Since $\operatorname{Gal} K^h = G^d(\widetilde{K}|K, v)$, we have that $v\iota a = v\sigma\iota_i\tau a = v\iota_i a$ for all $a \in L$, that is, $v\iota = v\iota_i$ on L. □

The above lemma allows us to describe all extensions of v from K to L using the representatives $\iota_1, \ldots, \iota_s$ of the respective double cosets. In particular, the number of such distinct extensions is precisely s.

Note that the field $K^{h(v_i)} = \iota_i^{-1}K^h$ lies in the henselization $L^{h(v_i)} = L.\iota_i^{-1}K^h$ (the last equality follows from Equation (6.7)). Since ι_i sends $\iota_i^{-1}K^h$ onto K^h and $L.\iota_i^{-1}K^h$ onto $\iota_i L.K^h$, we find that

$$[L^{h(v_i)} : K^{h(v_i)}] = [\iota_i L.K^h : K^h] = [(\iota_i L)^h : K^h]. \tag{6.8}$$

We can then apply Equation (6.5) to obtain:

$$[L : K] = \sum_{1 \le i \le s} [L^{h(v_i)} : K^{h(v_i)}]. \tag{6.9}$$

The degrees $[L^{h(v_i)} : K^{h(v_i)}]$ are called *local degrees*. Hence the equation says that the degree $[L : K]$ is the sum of the associated local degrees.

Finally, we will present a result on the behavior of the following ramification theoretical invariants related to polynomials that are close to each other. Recall from Section 1.4 that the ramification index of $(L|K, v)$ is $\mathrm{e}(L|K, v) = (vL : vK)$, and the inertia degree is $\mathrm{f}\,(L|K, v) := [Lv : Kv]$. The *Fundamental Inequality* (which can be found e.g. in [5]) states that

$$[L : K] \ge \sum_{1 \le i \le s} \mathrm{e}(L|K, v_i) \cdot \mathrm{f}\,(L|K, v_i).$$

If v extends uniquely from K to L, then by *the Lemma of Ostrowski* (see e.g. [13]) we have that

$$[L : K] = p^\nu \cdot \mathrm{e}(L|K, v) \cdot \mathrm{f}\,(L|K, v),$$

where $p = \operatorname{charexp} Kv$. The factor p^ν is called the *defect* of the extension $(L|K, v)$ and denoted by $\mathrm{d}(L|K, v)$.

In general, for a finite extension L of a valued field (K, v), we can define defects in a similar manner, using the fact that the valuation in each extension $L^{h(v_i)}|K^{h(v_i)}$ extends uniquely and that henselizations are immediate extensions:

$$\mathrm{d}(L|K, v_i) := \frac{[L^{h(v_i)} : K^{h(v_i)}]}{\mathrm{e}(L|K, v_i) \cdot \mathrm{f}(L|K, v_i)}. \tag{6.10}$$

This is often called the *Henselian defect*. We can now combine Equations (6.9) and (6.10), together with the definition of the defect, to obtain the following version of the Fundamental Inequality:

$$[L : K] = \sum_{1 \le i \le s} \mathrm{d}(L|K, v_i) \cdot \mathrm{e}(L|K, v_i) \cdot \mathrm{f}(L|K, v_i). \tag{6.11}$$

Lemma 6.3.4 ([9], Lemma 11.2)**.** *Let $\iota_1, \ldots, \iota_s$ and L be as in Notation 6.2.1. Then for each $1 \le i \le s$ we have that*

$$\begin{array}{rcccl} \mathrm{d}(L|K, v_i) & = & \mathrm{d}((\iota_i L).K^h|K^h, v) & = & \mathrm{d}(\iota_i L|K, v), \\ \mathrm{e}(L|K, v_i) & = & \mathrm{e}((\iota_i L).K^h|K^h, v) & = & \mathrm{e}(\iota_i L|K, v), \\ \mathrm{f}(L|K, v_i) & = & \mathrm{f}((\iota_i L).K^h|K^h, v) & = & \mathrm{f}(\iota_i L|K, v). \end{array}$$

Moreover, the following equality holds:

$$[L : K] = \sum_{1 \le i \le s} \mathrm{d}(\iota_i L|K, v) \cdot \mathrm{e}(\iota_i L|K, v) \cdot \mathrm{f}(\iota_i L|K, v). \tag{6.12}$$

Proof. Since henselizations are immediate extensions, we obtain:

$$\mathrm{f}(L|K, v_i) = [Lv_i : Kv_i] = [L^{h(v_i)}v_i : K^{h(v_i)}v_i] = [(L.\iota_i^{-1}K^h)v_i : (\iota_i^{-1}K^h)v_i].$$

As observed before, ι_i sends $\iota_i^{-1}K^h$ onto K^h and $L.\iota_i^{-1}K^h$ onto $\iota_i L.K^h$. Therefore, the above number is equal to

$$[(\iota_i L.K^h)v : K^h v] = [(\iota_i L)^h v : K^h v] = [\iota_i Lv : Kv] = \mathrm{f}(\iota_i L|K, v).$$

The result for $\mathrm{e}(L|K, v_i)$ is proved analogously from the same observations. The result for $\mathrm{d}(L|K, v_i)$ then follows by Equations (6.10) and (6.8). Those equalities together with Equation (6.11) imply Equation (6.12). □

The notions and results presented in the above lemmas now allow us to formulate the following root continuity theorem.

Theorem 6.3.5. *Let (K,v) be an arbitrary valued field, $f \in K[x]$ an irreducible monic polynomial over K and $\alpha \in \widetilde{K}$ a root of f. Further, let $v_1,\ldots,v_s$ be all extensions of v from K to $K(\alpha)$. Then there is some $\delta \in vK$ such that the following holds: If g is any irreducible monic polynomial over K satisfying $\operatorname{ins} g \geq \operatorname{ins} f$ and $v(f-g) > \delta$, and if $\beta \in \widetilde{K}$ is a root of g and $w_1,\ldots,w_t$ are all extensions of v from K to $K(\beta)$, then $s = t$ and after a suitable renumbering of the w_i, we have that*

$$\begin{aligned} \mathrm{d}(K(\alpha)|K,v_i) &= \mathrm{d}(K(\beta)|K,w_i) \\ \mathrm{e}(K(\alpha)|K,v_i) &= \mathrm{e}(K(\beta)|K,w_i) \\ \mathrm{f}\,(K(\alpha)|K,v_i) &= \mathrm{f}\,(K(\beta)|K,w_i)\,. \end{aligned}$$

Proof. Observe that by Lemma 6.3.3 the extensions of v from K to $K(\alpha)$ are in correspondence with the double cosets as in Notation 6.2.1 with $L := K(\alpha)$ via $v_i := v\iota_i$. By virtue of Lemma 6.2.4 we can choose the indices of the irreducible factors $f_1,\ldots,f_s$ of f over K^h in such a way that $\iota_i\alpha$ is a root of f_i. We do the same for g and its irreducible factors $g_1,\ldots,g_t$, taking $L := K(\beta)$ and choosing the automorphisms $\iota'_1,\ldots,\iota'_t$.

Take δ as in Theorem 6.2.5. Then the factors f_i and g_i satisfy the assertions of that theorem; in particular, we have that $s = r = t$. After a suitable renumbering, we can assume that $\iota'_i\beta$ is a root of g_i if and only if $\iota_i\alpha$ is a root of f_i. By Lemma 6.3.4 we have that

$$\mathrm{f}\,(K(\alpha)|K,v_i) = \mathrm{f}\,(\iota_i K(\alpha).K^h|K^h,v) = [(\iota_i K(\alpha).K^h)v : K^h v].$$

By Theorem 6.2.5, $\iota_i K(\alpha).K^h = K^h(\iota_i\alpha)$ and $\iota'_i K(\beta).K^h = K^h(\iota'_i\beta)$ are isomorphic over K^h. Therefore, the degree above is equal to

$$[K^h(\iota_i\alpha)v : K^h v] = [K^h(\iota'_i\beta)v : K^h v] = \mathrm{f}\,(\iota'_i K(\beta).K^h|K^h,v),$$

which in turn is equal to $\mathrm{f}\,(K(\beta)|K,w_i)$ by Lemma 6.3.4. The equations for the inertia degree are analogous. The result for the defect then follows from these equations, together with (6.8) and (6.10). □

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Tytuł rozprawy doktorskiej (**czcionka pogrubiona**): **Continuity of Roots and Values for Valued Fields**

promotor: stopień/tytuł naukowy/**imię i nazwisko** **prof. dr hab. Franz-Viktor Kuhlmann**
promotor pomocniczy: stopień/tytuł naukowy/ **imię i nazwisko -**
(jeżeli jest zatwierdzony uchwałą RW)

Streszczenie rozprawy doktorskiej w języku angielskim

We study connections between polynomials which are close to each other, i.e., whose respective coefficients are close in the topology induced by a valuation. The basic principle of root continuity states that the closeness of polynomials implies the closeness of their roots under a suitable pairing. This principle is then built upon to specify precise bounds for the ultrametric distances between the polynomials and their respective roots. Further, one finds additional connections between polynomials that are sufficiently close to each other. For example, the irreducible factors of such polynomials generate extensions which are isomorphic over a given ground field, and the respective extensions coming from the polynomials in question have the same ramification theoretical invariants. Furthermore, if two polynomials are close to each other, then their Newton Polygons coincide along a certain interval. We study this aspect of continuity in detail by employing the well-known result that connects the values of roots of a polynomial with the slopes of its Newton Polygon. This allows us to give precise statements on the values of roots of polynomials which are close to each other and leads to further results on root continuity in generality that was, to our knowledge, not seen before. The aforementioned statements can be adapted to continuity of roots and poles for rational functions with respect to a suitable extension of the ultrametric from the polynomial ring to the rational function field.

Data, podpis

30.03.2022 Hanna Ćmiel

słowa kluczowe w języku polskim (odpowiedniki słów kluczowych w języku angielskim).

waluacja (valuation), ciało waluacji (valued field), Wielokąt Newtona (Newton Polygon), ciągłość pierwiastków (continuity of roots), wielomiany nad ciałami waluacji (polynomials over valued fields), funkcje wymierne nad ciałami waluacji (rational functions over valued fields), przestrzeń ultrametryczna (ultrametric space),

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Tytuł rozprawy doktorskiej **(czcionka pogrubiona): Continuity of Roots and Values for Valued Fields**

Tłumaczenie tytułu rozprawy na język polski: **Ciągłość Pierwiastków oraz Wartości dla Ciał Waluacji**

promotor: stopień/tytuł naukowy/**imię i nazwisko** **prof. dr hab. Franz-Viktor Kuhlmann**
promotor pomocniczy: stopień/tytuł naukowy/ **imię i nazwisko** -
(jeżeli jest zatwierdzony uchwałą RW)

Streszczenie rozprawy doktorskiej w języku polskim

Badane są powiązania pomiędzy wielomianami, które są blisko siebie, tj. których współczynniki są blisko siebie w topologii indukowanej przez waluację. Podstawowa zasada ciągłości pierwiastków mówi, że bliskość wielomianów implikuje bliskość ich pierwiastków względem odpowiedniego doboru w pary. Na zasadzie tej buduje się dalsze rezultaty poprzez sprecyzowanie wartości ograniczającej ultrametryczną odległość pomiędzy wielomianami oraz pomiędzy ich odpowiednimi pierwiastkami. Pomiędzy wielomianami, które są bisko siebie, istnieją dalsze powiązania. Na przykład, nierozkładalne czynniki tychże wielomianów generują izomorficzne rozszerzenia nad ciałem bazowym, a odpowiednie rozszerzenia generowane przez całe wielomiany posiadają te same niezmienniki z teorii ramifikacji. Ponadto, jeśli wielomiany są blisko siebie, to ich Wielokąty Newtona są identyczne na pewnym odcinku. Szczegółowo badamy ten aspekt ciągłości poprzez wykorzystanie znanego twierdzenia, które łączy wartości pierwiastków wielomianu ze stopniami nachylenia Wielokąta Newtona. Pozwala nam to na precyzyjne rezultaty na temat wartości pierwiastków wielomianów, które są blisko siebie, co z kolei implikuje rezultaty nieznane dotychczas w takiej ogólności, jak te wymienione w dysertacji. Powyższe twierdzenia mogą być zaadaptowane do ciągłości pierwiastków i biegunów funkcji wymiernych względem odpowiedniego rozszerzenia ultrametryki z pierścienia wielomianów do ciała funkcji wymiernych.

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