Review of the Ph.D. thesis
"Continuity of Roots and Values for Valued Fields"
by Hanna Ćmiel

The problem of localizing polynomial roots and estimating the distances between roots can be traced back at least to the works of Cauchy. One spin-off of this theory is the problem of root continuity. Intuitively, if two polynomials are "close enough" to each other, then the roots of one polynomial should lie "close" to the roots of the other polynomial. The dissertation under review aims to study to what extent this intuitive image is preserved in a valuative setup when the notion of closeness is defined in terms of an ultra-metric induced by a valuation.

The dissertation consists of six chapters. The first chapter, titled "Notation and background" has purely introductory character. Its primary purpose is to fix the notation and terminology used throughout the thesis. The second chapter contains some results on root continuity, whose proofs do not require the notion of the Newton polygon (introduced later). In particular, the author presents Theorems 2.1.3 and 2.1.8, which are valuation-theoretic counterparts of the classical formulation of the root continuity. A different approach to root continuity is to consider a convergent sequence or a net of polynomials and look at the sequences (or nets) of their respective roots. This problem is studied in Section 2.2 (see Theorems 2.2.2 and 2.2.5).

Chapter 3 serves to introduce the notion of Newton polygons, which is the principal tool used in the rest of the dissertation. The main results of the thesis are contained in chapters 4 through 6. In Chapter 4, the author shows that if two polynomials are close enough to each other, then their Newton polygons partialy coincide (see Theorem 4.1.1). From this fact, the author draws a series of conclusions concerning root continuity (see Theorems 4.1.5, 4.2.2, 4.2.5, 4.2.8, and 4.2.9). Next, Chapter 5 is devoted to the problem of root continuity of rational functions. Finally, in Chapter 6, the author goes one step further and shows some applications of the results developed in the thesis. Two applications that are undoubtedly worth mentioning are: the "continuity of factorization" of polynomials (see Theorems 6.1.3 and 6.2.5) and the fact that the ramification-theoretical invariants of two field extensions coincide when the extensions come from polynomials close enough to each other (see Theorem 6.3.5). In my personal opinion, this last result is the most interesting in the dissertation under review.

The majority of the results constituting the thesis are not genuinely new. They are mainly generalizations and strengthenings of the results previously known. Nonetheless, they are interesting and push the boundary of what is known ahead. In particular, the results of

H. Ćmiel are mostly effective in the sense that instead of an ε–δ formulation of root continuity, the results of the thesis give explicit formulæ for δ (compare for example Theorem 2.1.3 with [15, Theorem 30.26]). Another important fact is that the majority of the results in the dissertation are proved without assuming that the polynomials in question are separable, which assumption prevails in earlier papers. These facts show that the author possesses a solid background in valuation theory, good mathematical intuition, and the ability to conduct independent scientific research.

The dissertation is written in a clear style. It is well organized, and its structure is logical. The author uses a mature mathematical language. One very positive aspect of the thesis is a large number of well-thought examples. Not only do they serve the role of illustrating various terms and results in the thesis, but—what is more important—they show that some assumptions are indispensable and some bounds cannot be improved. It shows that the presented results are in some respects optimal.

I have a minor objection concerning only Example 4.2.7. Its intent is to show that the bound for $v(f-g)$ in Theorem 4.2.2 cannot be improved. While the example is correct, it depends on a particular arbitrary extension of the definition of the Krasner constant to the case when a polynomial has just a single root. The Krasner constant is a valuation-theoretic counterpart of the minimal distance between roots of a polynomial. These two notions are clear and well defined when the polynomial has at least two distinct roots. However, when the polynomial has just one root, in the literature, there are at least three different extensions of the definition in the classical (i.e., archimedean) case. Therefore, it would be more desirable if the example used a polynomial f with more than one root, as this is the case where there are no inconsistencies between definitions.

In my opinion, the weakest part of the dissertation is Chapter 5. The use of a—quite arbitrarily chosen—ultrametric u is rather disputable. Neither is it entirely natural, as it depends on specific representations of elements, nor does it lead to some profound results that would justify this choice. In particular, the result concerning polynomials cannot be deduced from the results concerning rational functions.

As it is a reviewer's duty, I have to mention also some minor flaws of the dissertation. First of all, sometimes, a specific term is used within the text before it is formally defined (e.g., *cofinal sets* are defined on page 22 but used already on page 9). It does not cause much problem since all these terms are standard and well known. Nonetheless, if the author decided to define the term, she should have done it before the first use.

Secondly, it should be pointed out that the equality $\min_{i\leq j\leq n} va_j = v\partial_i f$ on page 12 line -5 is false in general. Take for example $K=\mathbb{Q}$,

$f = x^4 + 1$ and let v be the 2-adic valuation. Then for $i = 1$ we have $v\partial_1 f = 2$, while $\min_{1\leq j\leq 4} va_j = 0$. Fortunately, this error is in fact harmless, as the above-mentioned false equality is completely unneeded to prove the inequality following it.

It should also be noted that the assumption of Theorem 2.2.2 (and Corollary 2.2.3 following it) that all the polynomials in a net have the same degree is probably superfluous. We assume that the net converges, hence the polynomials will eventually have the degree equal to the degree of the limit.

All in all, my general opinion of the thesis is absolutely positive. As pointed out above, the dissertation contains several significant new mathematical results and constitutes an original solution to a scientific problem as required by Art. 187 pt. 2 *Ustawa z dnia 20 lipca 2018 Prawo o szkolnictwie wyższym i nauce*. The thesis proves that Hanna Ćmiel possesses general theoretical knowledge of mathematics and is capable of performing independent scientific research. Consequently, I postulate to admit Hanna Ćmiel to the next stage of the Ph.D. procedure.

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