**Referee's report on PhD dissertation**
***Algebraic hyperstructures in the model theory of valued fields***
**by mgr. Alessandro Linzi**

The PhD thesis under review deals with the model theory of valued fields from the point of view of the theory of (valued) *hyperfields*. The notion of a hyperfield generalizes the notion of a field by allowing the values of addition to be non-empty sets rather than just singletons. Krasner introduced this notion in 1957 having in mind exactly the context of valued fields (although, his motivations were algebraic and not model-theoretic).

Model theory of valued fields has been extensively studied for (at least) last 60 years and has numerous applications, for example: by Denef (rationality of certain Poincaré series) and by Hrushovski (motivic integration, non-standard Frobenius). This PhD thesis is about relative quantifier elimination in the context of valued fields. This line of research has also quite a long history starting from the work of Ax-Kochen and Ershov, then Macintyre (power predicates), Pas (angular component map), Basarab (mixed structures), Kuhlmann (amc structures), and Flenner (RV-structures). The thesis builds on the approaches of Kuhlmann and Flenner, and it also relates many of the above structures to the context of valued hyperfields.

The dissertation has five chapters and two appendices, which are discussed below where I also point out the main results of the thesis and add some other comments.

The first chapter gives a concise introduction to the model-theoretic notions and methods required for the rest of the thesis. It also introduces the first-order languages needed later for the model-theoretic treatment of (valued) hyperfields. A ternary relation symbol is chosen to represent the multivalued addition following the approach of Lee (I will comment more on it in the next paragraph).

The second chapter deals with the algebraic theory of valued hyperfields. One of the differences between the classical algebraic structures and the hyperstructures is that in the latter case one has to choose between the notions of "subset" and "equality" while trying to find proper general definitions (this distinction clearly vanishes while dealing with singletons). This is why there are many versions of possible axioms of hypergroups or hyperrings in the literature and one has to carefully choose the "proper" ones, which the author does. Similar problems occur with the notion of a morphism and, as the author correctly notices, equality (as in the (HH3") axiom on page 28) would be a bad choice here, since, for example, in a saturated hyperfield the addition would produce very large value sets and no small hyperfield would embed (with respect to the "(HH3")-notion" of a morphism) in a saturated one, which would be strange and undesired. It should be noted here that the main natural construction providing hyperfields comes from multiplicative quotients of fields. No other constructions producing hyperfields from the classical algebraic structures are mentioned in the thesis.

The third chapter provides links between valued fields and the algebraic hyperstructures described in the second chapter. To any valued field, an inverse system of valued hyperfields (indexed by the value group) is associated and these hyperfields are coming from certain multiplicative quotients discussed in Chapter 2. From the point of view of model theory, choosing a relational symbol to represent a hyperoperation has the obvious drawback that the model-theoretic substructures allow empty values of the hyperoperation, hence they need *not* be algebraic substructures. This problem looks impossible to overcome, but besides that it looks like the author manages to find a nice class of hyperfields where model-theoretic substructures satisfying this extra "non-emptyness" property mentioned above are hyperfields. This is Theorem 3.29 in the thesis and it is made more formal (to some extend) in Appendix B. The crucial property of this special class of hyperfields is that the value sets of addition are ultrametric balls, so "non-empty intersection" becomes "inclusion" and this is exactly what gives the model-theoretic simplification described above.

The fourth chapter deals with a more detailed study of the connections between the valued hyperfields associated to valued fields and the structures listed above such as: RV-structures, amc structures and angular component maps. Additionally, graded rings are considered as well in this context. All of this is needed for the applications to the model-theoretic results of the next and final chapter.

The fifth chapter contains the main results of the thesis, which are about the relative quantifier elimination for henselian valued fields of characteristic 0. Theorem 5.5 is a general result rephrasing a quantifier elimination theorem of Kuhlmann in terms of valued hyperfields. Using Theorem 5.5, the author gives in Theorem 5.11 a relative quantifier elimination statement for equicharacteristic henselian valued fields of characteristic 0 with respect to *one* ("0-valued") valued hyperfield. In the case of mixed characteristic, Corollary 5.32 provides a relative quantifier elimination result with respect to *infinitely many* ("$n \cdot vp$-valued") valued hyperfields.

This thesis is well-organized and generally well-written. I still have several specific comments and suggestions, which are listed below.

(1) I would prefer to have the main results of the thesis clearly listed in the introduction with a precise account of "who showed what", that is: which results were obtained by the author alone, and which results were obtained jointly.
(2) page 4: I am not aware of the name "*strict* morphism" in the context of model theory. What is its origin?
(3) For the general part about hyperrings (Chapter 2.1), I would prefer to have more examples (there are none except the multiplicative quotient construction and the 3-element hyperfield from Example 2.10) rather then the detailed and not very complicated proofs.
(4) page 55: "one cannot ensure that associativity...": are there counterexamples here or it is just a feeling (that "one cannot")?
(5) page 83, the discussion about graded rings in model theory: I do not quite understand the first paragraph here. The author first considers graded rings in a language containing a unary relation symbol (in particular, this language is *not* the language of rings $\mathcal{L}_r$) and then he concludes that "an $\mathcal{L}_r$-*theory* whose models are exactly graded rings does not exist". Besides the languages which do not fit, I have a feeling that the author may be confusing here *being unable to prove a given property* with *proving that a given property does not hold*.
(6) Definition 5.1. I do not think that this definition is in its final proper form. Some specific comments are below.
    (a) Does $A'$ depend on $i$? The order of quantification suggests this and the notation suggests otherwise.

(b) The condition (TR) does not seem to be a part of the definition, it rather looks like a condition to be satisfied to ensure the relative substructure completeness. For example, if $A'$ *does* depend on $i$ and if $A = \bigcup_i A'_i$, then it seems to me that (TR) implies the relative substructure completeness.
(c) No explicit examples are provided in the thesis about how this set-up is used, for example: I could not see any specification what "$A'$" from Definition 5.1 could be in the situations considered. I would definitely like to see explicitly how this set-up is used in Corollary 5.32.

(7) Lemmas 5.2 and 5.4: Both of these results follow rather directly from Łoś Theorem, which implies the following general principle:
"*All definable constructions commute with ultraproducts*".
I will illustrate this principle briefly by using an example of the equivalence relation. For simplicity, I will denote by $(M_i)_{\mathcal{U}}$ the ultraproduct of the structures $(M_i)_{i\in I}$ with respect to the ultrafilter $\mathcal{U}$ on $I$. If $R_i$ are definable equivalence relations on $M_i$, then we have a natural isomorphism (saying that "the construction of the quotient by a definable equivalence relation commutes with ultraproducts"):
$$(M_i/R_i)_{\mathcal{U}} \cong (M_i)_{\mathcal{U}}/(R_i)_{\mathcal{U}},$$
because for each sequences $(m_i, m'_i \in M_i)_{i\in I}$, we have:
$$\{i \in I \mid M_i \models R_i\,(m_i, m'_i)\} \in \mathcal{U} \qquad \text{iff} \qquad (M_i)_{\mathcal{U}} \models (R_i)_{\mathcal{U}}\,((m_i)_{\mathcal{U}}, (m'_i)_{\mathcal{U}})$$
by Łoś Theorem. So, my feeling is that the proofs of Lemmas 5.2 and 5.4 just repeat the classical proof of (some special cases of) Łoś Theorem.

(8) Appendix B: I would prefer here to have a full formal statement about universality of the theory of hyperfields in the context of the hyperfields from Corollary B.2. My guess is that such a statement should be something as:
"*All axioms are universal except for the* $\forall x, y \exists z r_+(x, y; z)$*-axiom*".

The above comments are of minor nature and therefore they do not affect the following:

***Conclusion***

In my opinion, this PhD thesis demonstrates a good general theoretic knowledge of the candidate in the scientific discipline of mathematics and his ability to perform original research. Therefore, I am happy to **confirm my overall positive evaluation of the thesis, I recommend this thesis to be accepted, and I also recommend to allow mgr. Alessandro Linzi to undertake the next steps in his PhD procedure**.

Piotr Leszek Kowalski

Elektronicznie podpisany przez Piotr Leszek Kowalski
Data: 2022.05.19 12:58:35 +03'00'

Prof. dr hab. Piotr Kowalski
Instytut Matematyczny
Uniwersytetu Wrocławskiego