

Report on the PhD thesis

Alessandro Linzi

Algebraic hyperstructures in the model theory of valued fields

(May 23, 2022)

The submitted dissertation deals with model-theoretic aspects of valued hyperfields. Hyperfields (and, more generally, hyperrings) are particular cases of so-called *hyperstructures*, namely, algebraic structures in which at least one operation is multivalued. One may think that generalizing algebraic structures in this way is good only for the sake of generalizing. Surprisingly, this is totally false. As it happens, hypergroups, hyperrings and hyperfields are relevant in modern number theory and algebraic geometry, including tropical geometry. Hyperstructures received attention by some top-class mathematicians, including the Fields medalist Alain Connes. Model theory of hyperstructures can be studied by treating the multivalued operations as relations. One of the main topics addressed in the thesis is quantifier elimination, leading to substructure completeness. This is an important issue studied in model theory, algebra, and theoretical computer science.

The dissertation is partially based on two preprints

1. Katarzyna Kuhlmann, Alessandro Linzi, Hanna Stojalowska, *Orderings and valuations in hyperfields*, 17 pages, arXiv:2106.04978
2. Alessandro Linzi, Hanna Stojalowska, *Hypervaluations on hyperfields and ordered canonical hypergroups*, 9 pages, arXiv:2009.08954

Both manuscripts are joint works. One of the co-authors, Hanna Stojalowska, is another PhD student at the University of Szczecin. In my opinion, such collaborations are extremely valuable and show that the Candidate is capable of working with mathematicians who are at a similar career stage, possibly without a direct guidance of the supervisor (who is not a co-author here). Both preprints contain a rather detailed study of (hyper)valuations on hyperfields and related structures. Some of the results are presented in Chapter 2.

Chapter 3 contains further study of valuations on hyperfields, in particular extending a recent work of Tolliver.

At the beginning of Chapter 4 the Candidate mentions some joint work with P. Błaszkiwicz, another young PhD(?) student, however no details of this work are given. Chapter 4 contains original results towards the relative quantifier elimination.

This is continued in Chapter 5, where the main result (Theorem 5.5) uses the technique of ultrapowers. Among applications, the author obtains substructure completeness of the theory of henselian valued fields of mixed characteristic (Corollary 5.32). Needless to say, Chapter 5 (almost 30 pages long) contains several more new results around the relative quantifier elimination.

The dissertation finishes with two appendices containing further explanation of the relevant concepts.

Summarizing, the main part of the thesis consists of Chapters 3 – 5, where one can find several original results extending the known theory of valued hyperfields, especially their model-theoretic aspects. The main theorems are definitely nontrivial, often require combining more than one proof technique. I hope the Candidate is preparing at least one more manuscript containing new results, especially those on (relative) quantifier elimination.

Critical remarks. I have to emphasize that the dissertation is written very carefully, it is really hard to find any misprints. The topic is important and worth further study.

My main criticism is the lack of clear statements concerning the Candidate’s achievements. Often it is hard to guess which results are really new and which are just combinations or rephrased versions of known results. The author’s results should have been emphasized in a clear way. Furthermore, the dissertation is a bit too long (over 130 pages). Perhaps some of the material could have been converted to another appendix, making the whole text more reader friendly.

Some more concrete remarks:

1. Page 3 (15). *Morphism* is the basic notion of category theory, while in model theory and algebra a standard name is *homomorphism*.
2. Page 23 (35). The name *Groupoid* is mostly reserved for categories whose all morphisms are isomorphisms, or algebraic structures with a partial associative and locally invertible operation. A set with an arbitrary binary operation is called a *magma* (although the word *groupoid* used to be common some years ago). Thus, the author could have used the name *hypermagma* or at least make a warning that groupoids have two totally different meanings.
3. Page 50 (62). Besides valuation theory, the notion of an ultrametric is *always* considered with the more natural ordering, opposite to the one used in Definition 3.20. I don’t mind defining the ultrametric upside down, however a comment on it is necessary.
4. Page 63 (75). The author writes “These sections contain joint work with P. Błaszkiwicz.” but there is no reference. Is it a work in progress?

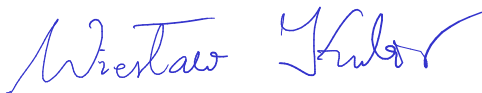
5. Pages 121 – 124 (Bibliography). Items [8], [51] are presented incorrectly. Moreover, I checked that these preprints are available online: <https://math.usask.ca/fvk/CoR.pdf> and <https://arxiv.org/abs/1811.08756>. Finally, as mentioned in the previous remark, there seems to be a joint work of A. Linzi with P. Błaszczewicz that does not appear in the bibliography at all.

Let me add that the remarks above are not relevant when it comes to my overall positive opinion on the value of this dissertation.

Conclusion. The PhD thesis of Alessandro Linzi contains new and nontrivial results in a pretty active and important area of pure mathematics. Without any hesitations, I recommend admitting the Candidate to the public defense.

More formally: The submitted thesis fulfills the conditions set by Article 187 ust. 1-3 ustawy z dnia 20 lipca 2018 r. Prawo o szkolnictwie wyższym i nauce (Dz. U. z 2022 r. poz. 574 ze zm.).

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