UNIVERSITY OF SZCZECIN

INSTITUTE OF MATHEMATICS

BARTOSZ WASILEWSKI

Asymptotics analysis and analytic tools for certain types of C_0-semigroups

SUPERVISOR: PROF. GRIGORIJ SKLYAR
AUXILIARY SUPERVISOR: DR. PIOTR POLAK

SZCZECIN 2023

OŚWIADCZENIE DOKTORANTA

Oświadczam, że moja praca pt.: „Asymptotics analysis and analytic tools for certain types of C_0-semigroups”

a. została napisana samodzielnie,
b. nie narusza praw autorskich w rozumieniu ustawy z dnia 14 lutego 1994 roku o prawie autorskim i prawach pokrewnych (Dz.U. 2019 r. poz. 1231) oraz dóbr osobistych chronionych prawem,
c. nie zawiera danych i informacji, które uzyskałem w sposób niedozwolony,
d. nie była podstawą nadania tytułu naukowego lub zawodowego ani mnie ani innej osobie.

Ponadto oświadczam, że treść pracy przedstawionej przeze mnie do obrony, zawarta na przekazanym nośniku elektronicznym jest identyczna z jej wersją drukowaną.

Szczecin, dn. ....

podpis doktoranta

OŚWIADCZENIE

Wyrażam zgodę na udostępnienie mojej pracy doktorskiej pt.: „Asymptotics analysis and analytic tools for certain types of C_0-semigroups”.

Szczecin, dn. ....

podpis doktoranta

mgr Bartosz Wasilewski

Tytuł rozprawy doktorskiej: **Asymptotics analysis and analytic tools for certain types of C_0-semigroups**

promotor: **prof. dr hab. Grigorij Sklyar**
promotor pomocniczy: **dr Piotr Polak**

Streszczenie rozprawy w języku polskim

Poniższa dysertacja została napisana z zamiarem zbadania własności asymptotycznych pewnej klasy nieograniczonych C_0-półgrup oraz, niezależnie, rozszerzenia istniejących wyników dotyczących równań różniczkowych z opóźnieniem typu neutralnego z przypadku $\mathbb{C}^n$ na przypadek nieskończeniewymiarowy

W pierwszej części dysertacji zostało udowodnione, że dla C_0-półgrupy $\{T(t)\}_{t\geq 0}$ o generatorze A, przy pewnych założeniach, zachodzi

$$\lim_{t\to\infty}\frac{\|T(t)A^{-1}\|}{f(t)}=0,$$

gdzie funkcja rzeczywista $f(t)$ jest w pewnym sensie podobna do normy półgrupy $\|T(t)\|$. Założenia dotyczą zachowania asymptotycznego obcięć półgrupy do pewnych rzutów Riesza stowarzyszonych z operatorem A. Dla odpowiednio regularnych C_0-półgrup, funkcja $f(t)$ może być równa $\|T(t)\|$. W tym wypadku uzyskany wyniki oznacza, że rozwiązania klasyczne odpowiedniego zagadnienia Cauchy'ego rosną wolniej (albo zanikają szybciej) niż norma półgrupy. Opisane wyniki poszerzają już istniejące, głównie poprzez dopuszczanie lokalizacji widma na osi $\{z\in\mathbb{C}:\ \mathrm{Re}(z)=\omega_0\}$.

W drugiej części dysertacji rozważane jest równanie różniczkowe

$$\dot{z}(t)=A\dot{z}(t-1)+\int_{-1}^{0}A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta+\int_{-1}^{0}A_3(\theta)z(t+\theta)\mathrm{d}\theta,\ z(t)\in H$$

gdzie H jest dowolną ośrodkową przestrzenią Hilberta a $A, A_2(\theta), A_3(\theta)$ są operatorami ograniczonymi o pewnych szczególnych własnościach. Opisane wyniki poszerzają wyniki już istniejące które zachodzą dla przypadku skończeniewymiarowego. Tymi wynikami są, między innymi, generowanie C_0-półgrupy poprzez operator liniowy $\mathcal{A}$ reprezentujący powyższe równanie w przestrzeni $H\times L^2\left([-1,0];H\right)$ oraz istnienie bazy Riesza skonstruowanej przy użyciu rzutów Riesza operatora $\mathcal{A}$.

....

mgr Bartosz Wasilewski

Title of the dissertation: **Asymptotics analysis and analytic tools for certain types of C_0-semigroups**

Supervisor: **Prof. Grigorij Sklyar**
Auxiliary supervisor: **Dr. Piotr Polak**

Dissertation summary

The object of this study was the analysis of asymptotic behavior of a certain class of unbounded C_0-semigroups and, independently, the extension of some existing results concerning the delay differential equations of the neutral type in $\mathbb{C}^n$ to the infinite-dimensional case.

In the first part of the dissertation, we prove that for the C_0-semigroup $\{T(t)\}_{t\geq 0}$ with the generator A having some particular asymptotic properties when truncated to the images of the Riesz projections of the operator A associated with certain subsets of the spectrum,

$$\lim_{t\to\infty} \frac{\|T(t)A^{-1}\|}{f(t)} = 0.$$

The real function $f(t)$ in some sense approximates the norm of the semigroup $\|T(t)\|$ and, for regular enough C_0-semigroups, the function $f(t)$ can equal $\|T(t)\|$. This property means that the classical solutions of the corresponding Cauchy problem grow slower (or decay faster) than the norm of the semigroup. Our results extend some existing ones, mainly by allowing the spectrum of the generator to be located on the the axis $\{z \in \mathbb{C} : \operatorname{Re}(z) = \omega_0\}$.

In the second part of the dissertation we consider the differential equation

$$\dot{z}(t) = A\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\mathrm{d}\theta,\ z(t) \in H$$

where H is an arbitrary separable Hilbert space and $A, A_2(\theta), A_3(\theta)$ are bounded linear operators with some particular properties. We extend the results which hold for the finite-dimensional case including the generation of a C_0-semigroup by the linear operator $\mathcal{A}$ representing the above equation and the existence of a Riesz basis of the corresponding space $H \times L^2([-1,0]; H)$ constructed from the Riesz projections of the operator $\mathcal{A}$.

....

Słowa kluczowe w języku polskim (odpowiedniki słów kluczowych w języku angielskim)/Keywords in Polish (English):

Asymptotyka C_0-półgrup operatorów (asymptotics of C_0-semigroups of operators), bazy Riesza podprzestrzeni (Riesz bases of subspaces), operatory Hilberta-Schmidta (Hilbert-Schmidt operators), przestrzenie Bochnera (Bochner spaces), równania różniczkowe z opóźnieniem typu neutralnego (delay differential equations of the neutral type), rzuty Riesza (Riesz/spectral projections).

Preface

The roots of the C_0-semigroup theory can be traced to the work of G. Peano from the end of the 19th century to whom we owe the exponential formula for the solution of the finite-dimensional time-dependent linear equation of the form

$$\dot{x}(t) = Ax(t),$$

and his student M. Gramegna extended the exponential formula to the case of bounded operators on infinite-dimensional spaces. While these formulas are quite simple, physical sciences of the first half of the 20th century needed more than that, namely solutions to ordinary linear differential equations for the case of the operator defining the differential equation being an unbounded operator. The answer to this need required new ideas in solving newly posed problems and thus originated the C_0-semigroup theory. The theory of C_0-semigroups is regarded as of now as the way to treat such linear ordinary differential equations in general Banach spaces. It became a well-established branch of functional analysis around the middle of the 20th century with the works of E. Hille, G. Lumer, R. Phillips, K. Yosida and others, who characterized the dynamical systems in Banach spaces that can be represented using C_0-semigroups. Classical examples of systems which can be described by the C_0-semigroup theory are partial differential equations, integro-differential equations, delay differential equations with quantum mechanics, population dynamics and control theory being less abstract examples. One can pose the question whether every *decent*, i.e., uniquely solvable ordinary linear differential equation in a Banach space, admits a semigroup representation. The answer is no, however the only additional condition that needs to hold in such a case is the non-emptiness of the semigroup generator's resolvent set. That being said, the C_0-semigroup theory is a powerful tool used for describing the dynamics of physical systems, which is a consequence of the fact that it finds application to linear one-parameter dynamical systems in any, no matter how abstract, Banach space[1]. As of now, the theory of C_0-semigroups is a very well developed, mature so to speak, field of knowledge. However there are still non-trivial questions left unanswered. An intensively studied field of research in the C_0-semigroup theory is the semigroups' asymptotic behavior. As one of the cornerstone results in C_0-semigroup stability one should consider the theorem given by [1] [16] [34], which provided the necessary and sufficient spectral condition for a bounded semigroup to be strongly stable, i.e., for all of its orbits to vanish with time. This result showed the qualitative difference in stability of finite-dimensional systems vs infinite-dimensional ones. This result has shown that infinite-dimensional systems can be strongly stable even if they have spectrum located on the imaginary axis, while for the finite-dimensional case,

[1] One can note here that it is also a consequence of the fact that the physical world can, for some reason, be described using Banach spaces.

in order for the system to be strongly stable, the matrix's spectrum needs to lie in the open left half-plane. However, whenever an infinite-dimensional system is strongly stable and the growth bound ω_0 is equal 0, due to the uniform boundedness principle, it cannot be uniformly stable. Due to [2], we know that this semigroup can be *semi-uniformly* stable, i.e, the smooth solutions can decay uniformly up to the multiplication by a constant. This idea is extended to the case of unbounded semigroups in [20]. In such a case, the assertion can, for regular enough semigroups, take the following form: The classical (smooth) solutions of a given Cauchy problem in a Banach space grow slower (or decay faster), than the norm of the semigroup. We will call such a semigroup *relatively stable*. Both of these results require the intersection of the generator's spectrum with the imaginary axis to be empty (we are considering the case of $\omega_0 = 0$ for simplicity). The first of the results presented in this work generalize the mentioned results from [2] [20]. This is achieved by showing that for a semigroup to be *relatively stable*, a more general condition, which allows for the spectrum of the semigroup's generator to be located on the imaginary axis, is sufficient. It is done by analyzing the asymptotics of the real function $t \to \|T(t)R_\mu\|$, where R_μ denotes the resolvent operator $(A-\mu)^{-1}$ at an arbitrary point μ belonging to the resolvent set $\rho(A)$ and A is the generator of the C_0-semigroup $\{T(t)\}_{t\geq 0}$. Here we prove that $\lim\limits_{t\to\infty} \frac{\|T(t)R_\mu\|}{f(t)} = 0$ for $f(t)$ similar in some sense (or equal to) the norm of the semigroup $\|T(t)\|$ whenever the behavior of the C_0-semigroup truncated to images of Riesz projections corresponding to spectrum located on the imaginary axis has better asymptotics than the function $f(t)$. We also provide examples of application to unbounded semigroups with the generator's spectrum located on the imaginary axis for which one can take $f(t) \equiv \|T(t)\|$. These examples first appeared, although in a different context, in [29] and [30]. This summarizes the first part of the results given in this dissertation.

In Chapter 3, which is the remaining part of this work, we analyze the infinite-dimensional delay systems of the form

$$\dot{z}(t) = A\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\mathrm{d}\theta,\ z(t) \in H, \tag{1}$$

where H is a separable Hilbert space and $A, A_2(\theta), A_3(\theta)$, $\theta \in [-1, 0]$, are bounded linear operators with some particular properties. The study of delay systems, for the finite-dimensional case, can be traced back to the works of such mathematicians as R. Bellman, N. Krasovskii, A. Myshkis from the middle of the 20th century. Since for such systems the initial condition is a function (on $[-1, 0]$ in the case of (1)), i.e, an infinite-dimensional object, it is only natural to try to model a delay system using the semigroup theory within the framework of infinite-dimensional Banach spaces. For the system (1) this can be done for both the finite [25] and infinite-dimensional cases. We prove the latter in this work using, similarly as in [25], the following representation of (1) in appropriate product Hilbert spaces:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix}, \quad \mathcal{A}\begin{pmatrix} y \\ z(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{z}(\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(\theta)\mathrm{d}\theta \\ \mathrm{d}z(\theta)/\mathrm{d}\theta \end{pmatrix}, \tag{2}$$

where $z_t(\cdot) = z(t+\cdot)$. The domain of the operator $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{(y, z(\cdot)) : z \in H^1([-1,0]; H), y = z(0) - Az(-1)\} \subset H \times L^2([-1,0]; H),$$

where H^1 denotes the Sobolev space of order 1. This type of representation for delay system was introduced by Burns et al. in [7]. Further in this chapter we focus on the the existence of a Riesz basis of subspaces which are invariant under the action of operator $\mathcal{A}$. It is a concept related to asymptotic stability of a given linear differential system. The Riesz basis property for the system (2) in the finite-dimensional case was obtained in [25] and was a key tool in analyzing the stability of the system (2) for the finite-dimensional case in [22–25]. The Riesz basis property occurs in a more general setting as was later proved in [37] [39]. Our results concerning the existence of such a Riesz basis of $\mathcal{A}$-invariant subspaces extend the results from [25] and are also somewhat similar to ones presented [37] [39], however the invariant subspaces that appear in this work are, in contrast to [25] [37] [39], infinite-dimensional. We prove the existence of a Riesz basis of infinite-dimensional $\mathcal{A}$-invariant subspaces for the system (2) for the case of $A_{2,3}(\cdot) \equiv 0$ and a weaker yet similar result for the operator-valued functions $A_{2,3}(\cdot)$ of a certain class which generalizes the matrix-valued functions used in the case of $H = \mathbb{C}^n$ in [25]. These results are applicable, among other, to integro-differential equations in the $L^2[0,1]$ space of the form

$$\dot{z}(s,t) = A\dot{z}(s,t-1) + \int_{-1}^{0}\int_{0}^{1} k_2(s,u,\theta)\dot{z}(u,t+\theta)\mathrm{d}u\mathrm{d}\theta + \int_{-1}^{0}\int_{0}^{1} k_3(s,u,\theta)z(u,t+\theta)\mathrm{d}u\mathrm{d}\theta,$$

under the condition

$$\int_{0}^{1}\int_{0}^{1} |k_{2,3}(s,u,\theta)|^2 \,\mathrm{d}u\mathrm{d}s < \infty$$

for all $\theta \in [-1,0]$, and

$$\int_{-1}^{0}\int_{0}^{1}\int_{0}^{1} |k_{2,3}(s,u,\theta)|^2 \,\mathrm{d}u\mathrm{d}s\mathrm{d}\theta < \infty.$$

Acknowledgments

I would like to thank Prof. Grigory Sklyar for taking care of my scientific development in a sense much wider than only the study and research in mathematics. I am also very grateful to Dr. Piotr Polak for the patience in helping me understand the subtleties and trivialities I have had a problem with more often than I would like to mention. I want to thank Dr. Jarosław Woźniak and Dr. Mateusz Firkowski for inspiring discussions and friendly help. What might sound unusual, I thank Mariusz Urbanek, the author of the book *"Genialni – Lwowska Szkoła Matematyczna"* (*Geniuses – the Lviv School of Mathematics*). The lecture of this book was the decisive factor of me doing research in the field of functional analysis, not to mention mathematics itself. Last but not least, I want to thank my family for providing me with a intellectually-stimulating environment and, most of all, my Father for teaching me the importance of disciplined thinking.

Contents

Chapter 1

Functional analytic tools

1.1 Elements of spectral and C_0-semigroup theory

In this section we recall some properties and definitions concerning C_0-semigroups and the spectrum and resolvent operator of linear operators acting from a Banach space X onto itself. The facts and definitions in this section are presented as in [10], unless noted. They can be found in many other general works on C_0-semigroup theory, such as [19] or [38]. First, for A being a closed linear operator on X with $D(A)$ denoting the domain of A we denote the resolvent set of A by $\rho(A)$ and its spectrum by $\sigma(A)$. Elements of the spectrum of the operator A can be classified in many different manners, here we will use the simplest characterization which splits the spectrum into the *approximate point spectrum*, which contains the *point spectrum*, and the *residual spectrum*.

Definition 1. *For a closed operator $A : D(A) \subseteq X \to X$, we call*

$$P\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda \textit{ is not injective } \}$$

the point spectrum *of A. Moreover, each $\lambda \in P\sigma(A)$ is called an* eigenvalue*, and each $0 \neq x \in D(A)$ satisfying $(A - \lambda)x = 0$ is an* eigenvector *of A (corresponding to λ).*

In the following definitions by $\mathrm{rg}(A)$ we mean the range of the operator A.

Definition 2. *For a closed operator $A : D(A) \subseteq X \to X$, we call*

$$A\sigma(A) := \begin{Bmatrix} \lambda \in \mathbb{C} : A - \lambda \textit{ is not injective or} \\ \lambda \in \mathbb{C} : \mathrm{rg}(A - \lambda) \textit{ is not closed in } X \end{Bmatrix}$$

the approximate point spectrum *of A.*

It is clear from the definition, that the point spectrum $P\sigma(A)$ is a subset of $A\sigma(A)$. The approximate point spectrum is characterized by the following property.

Lemma 3. *For a closed operator $A : D(A) \subseteq X \to X$ and a number $\lambda \in \mathbb{C}$ one has $\lambda \in A\sigma(A)$,* *i.e.,* λ *is an* approximate eigenvalue*, if and only if there exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ *called an* approximate eigenvector*, such that* $\|x_n\| = 1$ *and* $\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$.

The boundary of the spectrum belongs to the approximate point spectrum, i.e., the following holds

Lemma 4. *For a closed operator* $A : D(A) \subseteq X \to X$*, the topological boundary* $\partial\sigma(A)$ *of the spectrum* $\sigma(A)$ *is contained in the approximate point spectrum* $A\sigma(A)$*.*

The next definition followed by the Proposition 6 are useful when applying Theorem 10, which allows to determine some desirable asymptotic properties for semigroups of operators.

Definition 5. *For a closed operator* $A : D(A) \subseteq X \to X$ *we call*

$$R\sigma(A) := \{\lambda \in \mathbb{C} : \mathrm{rg}(A - \lambda) \textit{ is not dense in } X\}$$

the residual spectrum *of* A.

Proposition 6. *For a closed, densely defined operator* A*, the residual spectrum* $R\sigma(A)$ *coincides with the point spectrum* $P\sigma(A^*)$ *of* A^**, where* A^* *is the adjoint operator of the operator* A*.*

1.1.1 Asymptotics of C_0-semigroups

There are several different notions of stability of C_0-semigroups, the most important ones are listed below. Before we proceed however, we will recall for clarity the definition of the growth bound of a semigroup T, denoted by $\omega_0(T)$.

Definition 7. *Let* $T = \{T(t)\}_{t\geq 0}$ *be a* C_0*-semigroup. The* growth bound $\omega_0(T)$ *of* T *is defined as*

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} : \text{ there exists } M_\omega \geq 1 \text{ such that } \|T(t)\| \leq M_\omega e^{\omega t}, \text{ for all } t \geq 0\}$$

We will usually denote $\omega_0(T)$ shortly by ω_0. This number is always less than ∞, which follows from the observation that, due to the uniform boundness principle, $\|T(t)\|$ is uniformly bounded on all compact intervals. It can however equal $-\infty$ in some cases (so-called nilpotent C_0-semigroups).

Definition 8. C_0*-semigroup* $\{T(t)\}_{t\geq 0}$ *is called*

(a) uniformly exponentially stable if there exists $\varepsilon > 0$ *such that*

$$\lim_{t\to\infty} e^{\varepsilon t} \|T(t)\| = 0,$$

(b) uniformly stable if

$$\lim_{t\to\infty} \|T(t)\| = 0,$$

(c) strongly stable if

$$\lim_{t\to\infty} \|T(t)x\| = 0, \quad \text{for all } x \in X,$$

(d) *weakly stable if*

$$\lim_{t\to\infty} \langle T(t)x, x^*\rangle = 0, \quad \text{for all } x \in X \text{ and } x^* \in X^*,$$

where X^* denotes the dual space to the space X.

Note that a C_0-semigroup is uniformly exponentially stable (case (a)) if and only if ω_0 is less than 0. The following proposition characterizes in more detail the concept of uniform exponential stability.

Proposition 9. *For a C_0-semigroup $\{T(t)\}_{t\geq 0}$ the following assertions are equivalent.*

(a) *$\omega_0 < 0$, i.e., $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable.*

(b) $\lim_{t\to\infty} \|T(t)\| = 0.$

(c) *$\|T(t_0)\| < 1$ for some $t_0 > 0$.*

Now we will state a crucial theorem concerning strong stability of bounded C_0-semigroups.

Theorem 10. ([1] [16] [34]). *Let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X and let*

$$\sigma(A) \cap (i\mathbb{R}) \quad \textit{be at most countable,}$$

then the C_0-semigroup $\{T(t)\}_{t\geq 0}$ is strongly asymptotically stable *if and only if the operator A^* has no purely imaginary eigenvalues.*

1.1.2 Connection between C_0-semigroups and linear differential equations in Banach spaces

In this subsection we will show the connection between C_0-semigroups of operators and linear differential equations in Banach spaces. The abstract Cauchy problem (ACP) in a Banach space X, together with its classical solution, is defined as follows:

Definition 11. (a) *The initial value problem*

$$\begin{cases} \dot{x}(t) = Ax(t), & \text{for } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{ACP}$$

is called the abstract Cauchy problem *associated to $(A, D(A))$ and the initial value x_0.*

(b) *A function $x(t) : t \in \mathbb{R}_+ \to X$ is called a* (classical) *solution of* (ACP) *if $x(t)$ is continuously differentiable with respect to t, $x(t) \in D(A)$ for all $t \geq 0$, and* (ACP) *holds.*

Now, assume that $(A, D(A))$ is a generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Then the following holds:

Proposition 12. *Let $(A,(D(A))$ be the generator of the C_0-semigroup $\{T(t)\}_{t\geq 0}$. Then, for every $x \in D(A)$, the function*

$$x : t \to x(t) := T(t)x$$

is the unique classical solution of (ACP).

In many cases a more general concept of solution is of use, namely the *mild solution*, defined as:

Definition 13. *A continuous function $x(\cdot) : \mathbb{R}_0^+ \to X$ is called a* mild solution *of* (ACP) *if* $\int_0^t x(s)\mathrm{d}s \in D(A)$ *for all* $t \geq 0$ *and*

$$x(t) = A\int_0^t x(s)\mathrm{d}s + x_0.$$

Any classical solution of (ACP) of the form $T(t)x$ is also a mild solution. Next theorem will state the converse, in some sense, of Proposition 12.

Theorem 14. *Let $A : D(A) \subset X \to X$ be a closed operator. For the associated abstract Cauchy problem* (ACP) *we consider the following existence and uniqueness condition:*

$$\textit{For every } x_0 \in D(A), \textit{ there exists a unique solution } x(\cdot, x_0) \textit{ of (ACP)} \tag{EU}$$

Then the following properties are equivalent.

(a) A generates a C_0-semigroup.

(b) A satisfies (EU) *and $\rho(A) \neq \emptyset$.*

(c) A satisfies (EU)*, and there exist a sequence $\lambda_n \to \infty$ such that the ranges $(\lambda_n - A)D(A)$ equal X for all $n \in \mathbb{N}$.*

(d) A satisfies (EU)*, has dense domain, and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ satisfying $\lim_{n\to\infty} x_n \to 0$, one has $\lim_{n\to\infty} x(t, x_n) = 0$ uniformly in compact intervals $[0, t_0]$.*

Thus the existence of a unique solution of (ACP) combined with the non-emptiness of the resolvent set of the generator A are equivalent to the generation of a C_0-semigroup by the operator A.

We will now formulate the remarkable Hille-Yosida Theorem. This theorem shows the direct relationship between the exponential bound of the growth of the norm of the C_0-semigroup, localization of the set $\rho(A)$, and behavior of the norm of the resolvent on certain subsets of $\rho(A)$, where A is the generator of the C_0-semigroup.

Theorem 15. (Hille-Yosida Theorem) *Let $(A, D(A))$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$, $M \geq 1$ be constants. Then the following properties are equivalent,*

(a) (A,D(A)) generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying

$$\|T(t)\| \leq Me^{\omega t} \quad \textit{for } t \geq 0.$$

(b) (A,D(A)) is closed, densely defined, and for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and

$$\left\|\left[(\lambda - \omega)R(\lambda, A)\right]^n\right\| \leq M \quad \textit{for all } n \in \mathbb{N}.$$

(c) (A,D(A)) is closed, densely defined, and for every $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\lambda > \omega$ *one has* $\lambda \in \rho(A)$ *and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^n} \quad \textit{for all } n \in \mathbb{N}.$$

1.2 Riesz projections and Riesz bases of subspaces

Here we give some facts concerning the Riesz projections (spectral projections) and Riesz bases of subspaces (bases equivalent to an orthogonal base). The definitions and results in this section are presented as in [12]. The Riesz projections associate an operator with a contour integral of the resolvent in the complex plane, while the Riesz basis of subspaces is a family of subspaces in some sense close to an orthogonal basis of subspaces and only makes sense if we work in a Hilbert space.

Let X be a Banach space. The Riesz projection of the operator A corresponding to a curve Γ which is a subset of the resolvent set $\rho(A)$, denoted here by P_Γ, is defined as follows:

Definition 16. (Riesz Projection (Spectral Projection)) *Let* Γ *be a rectifiable simple or composite contour enclosing some region* G_Γ *and lying entirely in the resolvent set* $\rho(A)$ *of the operator* $A \in \mathcal{L}(X)$*. Then* $R(A, \lambda) = (A - \lambda)^{-1}$ *will be an analytic operator-valued function on* Γ*. Assume that the curve* Γ *has positive orientation relative to the region* G_Γ*, we then form the integral*

$$P_\Gamma = -\frac{1}{2\pi i} \int_\Gamma R(A, \lambda) \mathrm{d}\lambda,$$

then the following propositions take place

- *The operator* P_Γ *is a projection operator commuting with the operator* A *and hence in the decomposition*

  $$X = Y_\Gamma \oplus Z_\Gamma, \text{ where } Y_\Gamma = P_\Gamma X \text{ and } Z_\Gamma = (I - P_\Gamma)X$$

  both subspaces Y_Γ *and* Z_Γ *are invariant subspaces of the operator* A*. What is more,*

  (a) The spectrum of the restriction of the operator A *to the subspace* Y_Γ *is the part of the spectrum of the operator* A *contained in the region* G_Γ

  (b) The spectrum of the restriction of the operator A *to the subspace* Z_Γ *is the part of the spectrum of the operator* A *lying outside the closure of the region* G_Γ*,*

- *If* Γ_1 *and* Γ_2 *are two different contours having the properties indicated above and the regions* G_{Γ_1} *and* G_{Γ_2} *do not have common points, then the corresponding projectors are orthogonal to each other, i.e.,*

  $$P_{\Gamma_1} P_{\Gamma_2} = P_{\Gamma_2} P_{\Gamma_1} = 0.$$

Although the authors of [12] formulate Definition 16 for Hilbert spaces, they state in the introduction that "The first chapter recalls the well-known results general theory of bounded non-self-adjoint operators. Generally, these results are not specific to Hilbert space -they could be formulated for operators in a Banach space". We have decided to include the above definition of the Riesz projection due to its clarity. In chapter IV of [10] the authors provide a definition of the Riesz projection (spectral projection) which does not assume the space X to be a Hilbert space, only a Banach space, and also drop the requirement "$A \in \mathcal{L}(X)$" and require instead the operator A to be a closed linear operator.

The remaining part of this section is devoted to the concept of *Riesz basis of subspaces (basis of subspaces equivalent to an orthogonal basis)* in a Hilbert space, from now on denoted by H. We begin with the definition of a basis of subspaces of the space H followed by the definition of a Riesz basis of subspaces of the space H.

Definition 17. *A sequence $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ of nonzero subspaces $\mathfrak{M}_k \subset H$ is called a* basis of subspaces *of the space H, if any vector $x \in H$ decomposes uniquely in a series of the form*

$$x = \sum_{k=1}^{\infty} x_k,$$

where $x_k \in \mathfrak{M}_k$.

Definition 18. *A basis of subspaces for which the subspaces are mutually orthogonal is called an orthogonal basis of subspaces.*

Definition 19. (Riesz basis of subspaces) *Every bounded invertible operator $A : H \to H$ transforms any orthogonal basis of subspaces $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ of the space H to some other basis $\{\mathfrak{N}_k\}_{k=1}^{\infty}$ of the space H. A basis of subspaces $\{\mathfrak{N}_k\}_{k=1}^{\infty}$ obtained from an orthogonal basis with the use of such a transformation will be called* a Riesz basis of subspaces.

Now we state a necessary and sufficient condition for a sequence of subspaces to be a Riesz basis of subspaces.

Theorem 20. [11] *In order for a sequence $\{\mathfrak{M}_k\}_{k=1}^{\infty}$, which is a basis of subspaces of the space H, to be a Riesz basis of subspaces, it is necessary and sufficient that any permutation of its elements remains a basis of subspaces of the space H.*

Note that Theorem 20 implies that a Riesz basis of subspaces will remain one if we change the original norm $\|\cdot\|_1$ to an equivalent one $\|\cdot\|_2$. This can be seen by writing for any permutation $\sigma(k)$ of the indices k,

$$c\|x - \sum_{k=1}^{\infty} x_{\sigma(k)}\|_2 \le \|x - \sum_{k=1}^{\infty} x_{\sigma(k)}\|_1 \le C\|x - \sum_{k=1}^{\infty} x_{\sigma(k)}\|_2,$$

where $c, C > 0$, $x \in H$ is arbitrary and $x_{\sigma(k)}$ denote the elements of the representation of the element x with respect to the permuted basis in the space H endowed with the norm $\|\cdot\|_1$.

The following definitions are necessary to formulate Theorem 24, which gives a sufficient (and only sufficient) condition for a family of subspaces $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ to be a Riesz basis of subspaces.

Definition 21. *A sequence of subspaces $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ of the space H is said to be* complete *if the closed linear span of these subspaces is equal to the whole space H.*

Definition 22. *A sequence $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ of nonzero subspaces will be called* $\omega-$linearly independent *if* *the equality*

$$\sum_{k=1}^{\infty} x_k = 0, \quad (x_k \in \mathfrak{M}_k; k = 1, 2, ...)$$

cannot hold for

$$0 < \sum_{k=1}^{\infty} \|x_k\|^2 < \infty.$$

Definition 23. *Two sequences of subspaces* $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ *and* $\{\mathfrak{N}_k\}_{k=1}^{\infty}$ *are said to be* quadratically close *if*

$$\sum_{k=1}^{\infty} \|Q_k - P_k\|^2 < \infty,$$

where Q_k *and* P_k *are the orthogonal projections onto the subspaces* $\mathfrak{M}_k$ *and* $\mathfrak{N}_k$ *respectively.*

Now we are ready to provide a sufficient condition for a sequence of subspaces to constitute a Riesz basis. The Theorem is formulated for finite-dimensional subspaces and is extended to the infinite-dimensional case through Remark 26.

Theorem 24. [17] *A complete* $\omega-$*linearly independent sequence* $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ *of finite-dimensional subspaces which is quadratically close to some Riesz basis of subspaces* $\{\mathfrak{N}_k\}_{k=1}^{\infty}$ *of the space* H *basis is also a Riesz basis of subspaces.*

In order to formulate the extension to the infinite-dimensional case we will need the following definition of the *minimal angle* between subspaces.

Definition 25. *The minimal angle between the subspaces* $\mathfrak{U}$ *and* $\mathfrak{V}$ *is the angle* $\phi(\mathfrak{U}, \mathfrak{V})$ $(0 \leq \phi \leq \frac{\pi}{2})$, *defined by the equality*

$$\cos\phi(\mathfrak{U}, \mathfrak{V}) = \sup_{x \in \mathfrak{U}, y \in \mathfrak{V}, \|x\|=\|y\|=1} |\langle x, y \rangle|$$

The following remark extends Theorem 24 to the case of infinite-dimensional subspaces.

Remark 26. *In Theorem 24 we can discard condition of finite-dimensionality of subspaces (the proof remains the same) if we replace the condition of the* $\omega-$*linear independence of the sequence* $\{\mathfrak{M}_k\}_{k=1}^{\infty}$ *by the stronger condition: for any* k *the minimal angle between the subspace* $\mathfrak{M}_k$ *and the closed linear span of the rest of subspaces* $\mathfrak{M}_j$ $(j \neq k)$ *is positive.*

1.3 Bochner integral and Bochner spaces

The integrals that appear in this work are Bochner integrals. The Bochner integral is a generalization of the Lebesgue integral to the integral of functions taking value in Banach spaces. The facts and definitions in this section are presented as in [14], unless noted. Below we give some facts concerning the construction of the Bochner integral and its basic properties. We assume that a measure space $(S, \mathcal{F}, \mu)$ is given. By X we denote an arbitrary Banach space.

Definition 27. *A* μ-simple function *with values in* X *is a function of the form* $f = \sum_{n=1}^{N} \chi_{A_n} x_n$, *where* $x_n \in X$ *and the sets* $A_n \in \mathcal{F}$ *satisfy* $\mu(A_n) < \infty$ *and* χ_{A_n} *denotes the characteristic function of the set* A_n.

Definition 28. *A function* $f : S \to X$ *is strongly* μ*-measurable if there exists a sequence* $\{f_n\}_{n=1}^{\infty}$ *of* μ*-simple functions converging to* f μ*-almost everywhere.*

Definition 29. *For a* μ*-simple function* $f = \sum_{n=1}^{N} \chi_{A_n} x_n$ *we define*

$$\int_S f \mathrm{d}\mu := \sum_{n=1}^{N} \mu(A_n) x_n.$$

Now we are ready to state the definition of the Bochner integral.

Definition 30. (Bochner Integral) *A strongly* μ*-measurable function* $f : S \to X$ *is* Bochner integrable *w.r.t the measure* μ *if there exists a sequence of* μ*-simple functions* $f_n : S \to X$ *such that*

$$\lim_{n\to\infty} \int_S \|f - f_n\| \mathrm{d}\mu = 0.$$

Note that $s \to \|f(s) - f_n(s)\|$ *is* μ*-measurable, so that this definition makes sense. From*

$$\begin{aligned}\left\| \int_S f_n \mathrm{d}\mu - \int_S f_m \mathrm{d}\mu \right\| &\leq \int_S \|f_n - f_m\| \mathrm{d}\mu \\ &\leq \int_S \|f_n - f\| \mathrm{d}\mu + \int_S \|f_m - f\| \mathrm{d}\mu\end{aligned}$$

we see that the integrals $\int_S f_n \mathrm{d}\mu$ *form a Cauchy sequence. By completeness, this sequence converges to an element of* X*. This limit is called the* Bochner integral *of* f *with respect to* μ*, notation*

$$\int_S f \mathrm{d}\mu := \lim_{n\to\infty} \int_S f_n \mathrm{d}\mu.$$

The following basic properties of the Bochner integral will be of use in Chapter 3. Note that Theorem 32 is a generalization of the Dominated Convergence Theorem for the Lebesgue integral.

Proposition 31. *A strongly* μ*-measurable function* $f : S \to X$ *is Bochner integrable with respect to* μ *if and only if*

$$\int_S \|f\| \mathrm{d}\mu < \infty,$$

and in this case we have

$$\left\| \int_S f \mathrm{d}\mu \right\| \leq \int_S \|f\| \mathrm{d}\mu.$$

Theorem 32. (Dominated Convergence Theorem) *Let the functions* $f_n : S \to X$ *be Bochner integrable. If there exists a function* $f : S \to X$ *and a non-negative integrable function* $g : S \to \mathbb{R}$ *such that* $\lim_{n\to\infty} f_n = f$ *almost everywhere and* $\|f_n\| \leq g$ *almost everywhere, then* f *is Bochner integrable and we* *have*

$$\lim_{n\to\infty}\int_S \|f_n - f\|\mathrm{d}\mu = 0.$$

In particular,

$$\lim_{n\to\infty}\int_S f_n\mathrm{d}\mu = \int_S f\mathrm{d}\mu.$$

The Bochner integral allows to consider a generalization of the Banach $L^p(S,\mathbb{C})$ spaces of Lebesgue p-integrable functions. These Banach spaces are called *Bochner spaces* and are defined as follows.

Definition 33. (Bochner space) *For $1 \le p < \infty$ we define $L^p(S;X)$ as the linear space of all (equivalence classes of) strongly μ-measurable functions $f : S \to X$ for which*

$$\int_S \|f\|^p \mathrm{d}\mu < \infty.$$

We define $L^\infty(S;X)$ as the linear space of all (equivalence classes of) strongly μ-measurable functions $f : S \to X$ for which there exists a real number $r \ge 0$ such that $\mu\{s : \|f(s)\| > r\} = 0$.

Endowed with the norms

$$\|f\|_{L^p(S;X)} := \left(\int_S \|f\|^p \mathrm{d}\mu\right)^{\frac{1}{p}}$$

and

$$\|f\|_{L^\infty(S;X)} := \inf\left\{r \ge 0 : \mu\{\|f\| > r\} = 0\right\},$$

the spaces $L^p(S;X), 1 \le p \le \infty$, are Banach spaces.

Now we will state the natural definition of strongly μ-measurable operator-valued functions which allows to consider Bochner integrals of functions which take values in $\mathcal{L}(X,Y)$, where Y denotes an arbitrary Banach space.

Definition 34. *A function $f : S \to \mathcal{L}(X,Y)$ is called strongly μ-measurable if for all $x \in X$ the Y-valued function $fx : s \to f(s)x$ is strongly μ-measurable.*

For the strongly μ-measurable operator-valued functions the following holds

Proposition 35. *Let $(S,\mathcal{F},\mu)$ be a measure space and let X and Y be Banach spaces. If $f : S \to X$ and $g : S \to \mathcal{L}(X,Y)$ are strongly μ-measurable, then $gf : S \to Y$ is strongly μ-measurable.*

Below provide a result concerning the integration by parts of Bochner integrals.

Theorem 36. [9] *If $f(\cdot) : [a,b] \to X$ and $T(\cdot) : [a,b] \to \mathcal{L}(X)$ are Bochner integrable on $[a,b]$, then*

$$\int_a^b T(t)\left(\int_a^t f(s)\mathrm{d}s\right)\mathrm{d}t = -\int_a^b \left(\int_a^s T(t)\mathrm{d}t\right) f(s)\mathrm{d}s + \left(\int_a^b T(t)\mathrm{d}t\right)\left(\int_a^b f(s)\mathrm{d}s\right).$$

1.4 Hilbert-Schmidt operators

Here we define the Hilbert-Schmidt operators, a class of bounded operators with some very useful properties, including the fact that they form a Hilbert space for some appropriate scalar product. The facts presented here come from [35] and should be found in any basic textbook concerning the Hilbert space operator theory. First, we need the definition of the *trace* of a positive bounded operator.

Definition 37. *Let H be a Hilbert space with the orthonormal basis $\{e_n\}_{\in N}$. Let A be a positive bounded operator, we define the* trace *of A, denoted by $\mathrm{Tr}(A)$, as*

$$\mathrm{Tr}(A) = \sum_{i \in N} \langle Ae_i, e_i \rangle \in [0, +\infty]$$

Next proposition shows than the trace of an operator does not depend on the choice of the orthonormal basis.

Proposition 38. *Let $\{e'_i\}_{i \in N'}$ be a different orthonormal basis of H and let A be a positive operator. Define*

$$\mathrm{Tr}'(A) = \sum_{i \in N'} \langle Ae'_i, e'_i \rangle,$$

then $\mathrm{Tr}'(A) = \mathrm{Tr}(A)$.

Now we are ready to define the space of Hilbert-Schmidt operators $\mathcal{L}_{HS}(H)$.

Definition 39. (Hilbert-Schmidt Operator) *Let $A \in \mathcal{L}(H)$. We call A a* Hilbert-Schmidt operator*, whenever*

$$\mathrm{Tr}(A^*A) < \infty,$$

and denote the space of Hilbert-Schmidt operators as $\mathcal{L}_{HS}(H)$.

For operators in $\mathcal{L}_{HS}(H)$ the trace operation can be extended to a scalar product and the space $\mathcal{L}_{HS}(H)$ forms a Hilbert space, as is stated in the following Theorem.

Theorem 40. *$\mathcal{L}_{HS}(H)$ is a Hilbert space with the scalar product given by*

$$\langle A, B \rangle_{\mathcal{L}_{HS}(H)} = \mathrm{Tr}(B^*A) = \sum_{i \in N} \langle Ae_i, Be_i \rangle. \quad A, B \in \mathcal{L}_{HS}(H).$$

The norm given by this scalar product

$$\|A\|_{\mathcal{L}_{HS}(H)} = \left(\mathrm{Tr}(A^*A)\right)^{\frac{1}{2}}, \quad A \in \mathcal{L}_{HS}(H)$$

is called the Hilbert-Schmidt *norm.*

It can be seen from the definition of the trace of a positive operator that, for the space H being separable and for $A \in \mathcal{L}_{HS}(H)$, it holds that

$$\|A\|^2_{\mathcal{L}_{HS}(H)} = \mathrm{Tr}(A^*A) = \sum_{i\in\mathbb{N}} \|Ae_i\|^2 = \sum_{i,j\in\mathbb{N}} |\langle Ae_i, e_j\rangle|^2 = \sum_{i,j\in\mathbb{N}} |a_{ij}|^2, \tag{1.1}$$

where a_{ij} are the elements of the infinite matrix representing the operator A in an arbitrary orthonormal basis. Note that it follows from (1.1) that

Remark 41. *The space of Hilbert-Schmidt operators on a Hilbert space H is separable whenever H is separable.*

Also note that, as can be seen from (1.1), the Hilbert-Schmidt operators form an extension of matrix operators used for the case when $H = \mathbb{C}^n$. The Hilbert-Schmidt norm, whenever defined, dominates the standard operator norm, i.e.,

Proposition 42. *For any Hilbert-Schmidt operator A it holds that*

$$\|A\|_{\mathcal{L}(H)} \leq \|A\|_{\mathcal{L}_{HS}(H)},$$

where $\|\cdot\|_{\mathcal{L}(H)}$ denotes the standard operator norm.

1.4.1 Hilbert-Schmidt operators on $L^2([a,b],\mathbb{C})$

Here we introduce a class of integral operators acting on the space $L^2([a,b],\mathbb{C})$ which are Hilbert-Schmidt operators. Let $f(\cdot) \in L^2([a,b],\mathbb{C})$.

Proposition 43. *The operator A defined by*

$$(Af(\cdot))(t) = \int_a^b k(s,t)f(s)\mathrm{d}s,$$

is a Hilbert-Schmidt operator whenever

$$\int_a^b \int_a^b |k(s,t)|^2 \,\mathrm{d}s\mathrm{d}t < \infty,$$

with the Hilbert-Schmidt norm given by

$$\|A\|^2_{\mathcal{L}_{HS}(L^2[a,b])} = \int_a^b \int_a^b |k(s,t)|^2 \,\mathrm{d}s\mathrm{d}t.$$

1.5 Auxiliary tools

We will now state the definition of the Sobolev space of order one. In this dissertation we will not need the definition of higher-order Sobolev spaces, although the first equality in the definition below can be naturally extended to functions differentiable more than once.

Definition 44. [8] *Let X be a Banach space, fix a real number $1 \leq p < \infty$, and let $I = [a,b] \subset \mathbb{R}$ be a compact interval.* The Sobolev Space $W^{1,p}(I,X)$ *can be defined as the completion of the spaces of* *weakly differentiable functions $f(\cdot): I \to X$ with respect to the norm*

$$\|f(\cdot)\|_{W^{1,p}} := \left(\int_a^b \|f(t)\|^p + \|f'(t)\|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

Alternatively, $W^{1,p}(I,X)$ is the space of all functions $f(\cdot): I \to X$ that can be expressed as the integrals of L^p functions, i.e.

$$W^{1,p}(I,X) := \left\{ \begin{array}{l} f(\cdot): I \to X| \text{ there exists a strongly measurable function} \\ g(\cdot): I \to X \text{ such that } \int_a^b \|g(t)\|^p \, \mathrm{d}t < \infty \\ \text{ and } f(t) - f(a) = \int_a^t g(s) \mathrm{d}s \text{ for all } t \in I. \end{array} \right\}.$$

The Sobolev space $W^{1,p}(I,X)$ is a Banach space. Note the case of $p = 2$ and the space X being a Hilbert space, the space $W^{1,2}(I,X)$ is a Hilbert space and is denoted by $H^1(I,X)$.

The following is a classical result concerning invertibility of operators sufficiently close to invertible operators.

Theorem 45. [15] *If a linear operation $A \in \mathcal{L}(X)$ on a Banach space X has an inverse $A^{-1} \in \mathcal{L}(X)$, and the norm of the operation ΔA satisfies the inequality $\|\Delta A\|_{\mathcal{L}(X)} < \left\|A^{-1}\right\|_{\mathcal{L}(X)}^{-1}$, then the operation $A_\Delta = A + \Delta A$ has an inverse A_Δ^{-1} and the following inequality holds*

$$\left\|A_\Delta^{-1} - A^{-1}\right\|_{\mathcal{L}(X)} < \frac{\left\|A^{-1}\right\|_{\mathcal{L}(X)}}{1 - \|A^{-1}\|_{\mathcal{L}(X)} \|\Delta A\|_{\mathcal{L}(X)}}.$$

Below we present, probably the most elementary functional calculus for bounded operators, which is due to the work of N. Dunford.

Definition 46. [38] *Consider a bounded linear operator $A \in \mathcal{L}(X)$ where X is a complex Banach space. We define a function $f(A)$ of the operator A by a formula similar to the Cauchy integral formula:*

$$f(A) = -\frac{1}{2\pi i} \int_C f(\lambda) R(A, \lambda) \mathrm{d}\lambda$$

for $C \subset \rho(A)$, where $R(A, \lambda)$ denotes the operator $(A - \lambda)^{-1}$.

Theorem 47. [38] *Let $f(\lambda)$ belong to the family $F(A)$ of all complex-valued functions which are holomorphic in some neighborhood of the spectrum $\sigma(A)$ of the operator A, and let an open set $U \supset \sigma(A)$ of the complex plane be contained in the domain of holomorphy of $f(\lambda)$, and suppose further that the boundary ∂U of U consists of a finite number of rectifiable Jordan curves, oriented in positive sense. Then the bounded linear operator $f(A)$ will be defined by*

$$f(A) = -\frac{1}{2\pi i} \int_{\delta U} f(\lambda) R(A, \lambda) \mathrm{d}\lambda.$$

and the integral on the right may be called a Dunford's integral. By Cauchy's integral theorem, the value $f(A)$ depends only on the function f and the operator A, but not on the choice of the domain U. Then the following operational calculus *holds:*

If f and g are in $F(A)$, and α and β are complex numbers, then

- $\alpha f + \beta g \in F(A)$ *and* $\alpha f(A) + \beta g(A) = (\alpha f + \beta g)(A)$,
- fg *is in* $F(A)$ *and* $f(A)g(A) = (fg)(A)$.
- *if f has the Taylor expansion* $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *convergent in a neighborhood U of $\sigma(A)$, then* $f(A) = \sum_{n=0}^{\infty} \alpha_n A^n$ *(in the operator norm topology),*
- *let* $f_n \in F(A)$ $(n = 1, 2, ...)$ *be holomorphic in a fixed neighborhood U of $\sigma(A)$. If $f_n(\lambda)$ converges to $f(\lambda)$ uniformly on U, then $f_n(A)$ converges to $f(A)$ in the operator norm topology,*
- *if* $f \in F(A)$, *then* $f \in F(A^*)$ *and* $f(A^*) = f(A)^*$.

Chapter 2

On the relative decay of unbounded C_0-semigroups on the domain of the generator

The results presented in this chapter have been accepted for publication in the *Journal of Mathematical Physics, Analysis, Geometry* [33].

2.1 Introduction

The asymptotic behavior of C_0-semigroups and their orbits has been a subject of an intense study for the last few decades, see e.g. [3] [5] [6] [13] [36]. Due to the Theorem 10, the spectrum of the generator of a bounded C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ being located in the open left-half plane yields the semigroup T strongly stable. Due the uniform boundedness principle this stability cannot be uniform whenever $\omega_0(T) = 0$. Indeed, assume the contrary, i.e., there exist a positive function $g(t)$ such that
$g(t) \to 0$ as $t \to \infty$, and

$$\|T(t)x\| \leq g(t)\|x\|, \quad t \geq 0, \;\; x \in X.$$

Now, assuming that $g(t) \neq 0$ for all $t \geq 0$ we can restate the above as

$$\frac{\|T(t)x\|}{g(t)} \leq \|x\|, \quad t \geq 0, \;\; x \in X.$$

Which means that the set of vectors $\left\{\frac{T(t)x}{g(t)}\right\}_{t\geq 0}$ is bounded for all x and thus, by applying the uniform boundedness principle, we get that the set of non-negative numbers $\left\{\left\|\frac{T(t)}{g(t)}\right\|\right\}_{t\geq 0}$ is also bounded, i.e., $\|T(t)\| \leq Mg(t)$ for some $M > 0$, thus $\|T(t)\| < 1$ for t large enough. This implies, due to Proposition 9, that $\omega_0(T) < 0$, which is a contradiction. However, even though we do not have uniform stability in this case, due to [2] [4] there is the following theorem.

Theorem 48. *Let $T = \{T(t)\}_{t\geq 0}$ be a bounded C_0-semigroup acting on a Banach space X and let A be its generator. Then $\|T(t)A^{-1}\| \to 0$ as $t \to \infty$ if and only if the intersection of the spectrum of the generator A with the imaginary axis $\sigma(A) \cap (i\mathbb{R})$ is empty.*

The above means that for a bounded C_0-semigroup T for which

$$\sigma(A) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}, \tag{2.1}$$

the orbits starting in the domain of the generator are dominated uniformly up to the multiplication by a constant by a decaying function $f(t) = \|T(t)A^{-1}\|$, i.e. $\|T(t)x\| \leq f(t)C_x$ for all $x \in D(A)$, where $C_x = \|Ax\|$. This can be easily seen be writing $\|T(t)x\| = \|T(t)A^{-1}Ax\|$. With this being the case, we call the C_0-semigroup *semi-uniformly stable* [4]. Moreover, the semi-uniform stability may occur even for unbounded C_0-semigroups (see [32] for example). For the case of unbounded C_0-semigroups it was shown in [27] that the condition (2.1) remains necessary for $\|T(t)A^{-1}\| \to 0$. We note here that the sufficiency part of Theorem 48 for C_0-semigroups of contractions has been proved independently in [21]. The results obtained in [21] were later extended in [20] to obtain Theorem 49, which generalizes the sufficiency part of Theorem 48. Before we proceed to this results, we need to recall some necessary definitions.

$L^1_\alpha(\mathbb{R}_0^+)$ is the Banach algebra of functions for which

$$\|f\|_{L^1_\alpha(\mathbb{R}_0^+)} = \int_0^\infty |f(t)|\alpha(t)\mathrm{d}t < \infty$$

and the weight $\alpha(t)$ is nonquasianalytic when

$$\int_0^\infty \frac{\log \alpha(t)}{1+t^2}\mathrm{d}t < \infty.$$

For a nonquasianalytic weight $\alpha(t)$ the limit $\omega(\alpha) = \lim\limits_{t\to\infty} \frac{\log \alpha(t)}{t} = 0$ (it has *zero exponential type*, cf. [20]). The reduced weight function $\alpha_1(t) \equiv \limsup\limits_{s\to\infty} \frac{\alpha(t+s)}{\alpha(s)}$ inherits this property (again cf. [20])

A function $f \in L^1_\alpha(\mathbb{R}_0^+)$ is of spectral synthesis w.r.t. a closed subset Γ of $\mathbb{R}$ whenever there exists a sequence $f_n \in L^1_\alpha(\mathbb{R}_0^+)$ such that the Fourier transform of each f_n vanishes on an open neighborhood U_n of Γ for each n, and $\|f_n - f\|_{L^1_\alpha(\mathbb{R}_0^+)} \to 0$ as $n \to \infty$ (see [20] for a more detailed characterization).

Now we are ready to state the result from [20] which generalizes the sufficiency part of Theorem 48:

Theorem 49. [20] *Let $T = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup dominated by a weight function $\alpha(t)$ such that the corresponding reduced weight $\alpha_1(t)$ is nonquasianalytic. Assume that f is a function in $L^1_\alpha(\mathbb{R}_0^+)$ which is of spectral synthesis in the algebra $L^1_{\alpha_1}(\mathbb{R}_0^+)$ with respect to the set $\sigma(A) \cap (i\mathbb{R})$. Then*

$$\lim_{t\to\infty} \frac{1}{\alpha(t)} \left\| T(t) \int_0^\infty f(s)T(s)\mathrm{d}s \right\| = 0.$$

The above theorem implies the subsequent corollary:

Corollary 50. [20] *Let $T = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup dominated by a weight function $\alpha(t)$ such that the corresponding reduced weight $\alpha_1(t)$ is nonquasianalytic. Assume that the intersection of the spectrum of the generator A with the imaginary axis $\sigma(A) \cap (i\mathbb{R})$ is empty. Then*

$$\lim_{t\to\infty} \frac{1}{\alpha(t)} \left\| T(t) \int_0^\infty f(s)T(s)\mathrm{d}s \right\| = 0$$

for each f in $L^1_\alpha(\mathbb{R}^+_0)$. In particular,

$$\lim_{t\to\infty} \frac{1}{\alpha(t)} \|T(t)A^{-1}\| = 0. \tag{2.2}$$

If $\Gamma = \emptyset$, then any function in $L^1_\alpha(\mathbb{R}^+_0)$ is of spectral synthesis with respect to the set Γ. With the above in mind, it easy to see that by choosing $f(t) \equiv e^{-\lambda t}$ with $\lambda > 0$ large enough one can obtain (2.2). The use of this result relies however on $\sigma(A) \cap (i\mathbb{R}) = \emptyset$ Here we obtain an analogous result to Corollary 50, however allowing for the spectrum of the generator to be located on the imaginary axis. Moreover we prove that for sufficiently regular C_0-semigroups the following holds

$$\lim_{t\to\infty} \frac{\|T(t)R_\mu\|}{\|T(t)\|} = 0, \qquad \text{for } \mu \in \rho(A), \tag{2.3}$$

where by R_μ we mean the resolvent of the C_0-semigroup's generator A at the point $\mu \in \rho(A)$. For bounded C_0-semigroups (with $\omega_0(T) = 0$) the assertion (2.3) reduces to the sufficiency part of Theorem 48. Example 51 shows that the generalized condition

$$(\omega_0(T) + i\mathbb{R}) \cap \sigma(A) = \emptyset \tag{2.4}$$

is not necessary for (2.3) to hold for unbounded C_0-semigroups, although it is sufficient for a class of regular enough C_0-semigroups.

The papers [37] [39] provide an important tool for verification of (2.3) in Hilbert spaces whenever the spectrum of the generator A is discrete, and the eigenvalues are uniformly separated, (i.e., $\inf\{|\lambda_k - \lambda_m| : k, m \in \mathbb{N}, k \neq m\} > 0$), and the span of the corresponding eigenvectors is dense. For this being the case the eigenvectors will constitute a Riesz basis and the problem can be often reduced to solving it in the invariant subspaces. This approach clearly cannot be used for general Banach spaces. In this chapter we provide means for the verification of (2.3) for arbitrary Banach spaces. In the Section 2.3 we give an example of a family of unbounded C_0-semigroups for which (2.3) holds. For this family of semigroups it holds that $\sigma(A) \subset (i\mathbb{R})$, $\sigma(A)$ is countable, consists of simple eigenvalues only, however the eigenvectors do not constitute a Riesz basis

2.2 Main result

First we provide an example which shows that the condition (2.4) is not necessary for the property (2.3) to hold. For this example of an unbounded C_0-semigroup it holds that $\omega_0 = 0$ and that (2.3) holds, albeit $\sigma(A) \cap (i\mathbb{R}) \neq \emptyset$.

Example 51. *Consider a separable Hilbert space H with the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and put*

$$T(t)e_0 = e^{it}e_0, \quad T(t)e_{2k-1} = e^{(ik-\frac{1}{k})t}e_{2k-1}, \quad T(t)e_{2k} = e^{(ik-\frac{1}{k})t}(te_{2k-1}+e_{2k}),$$

for $k = 1, 2, \ldots$ The above defines a C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ on H. It is easy to see that on the invariant subspace

$$H_1 = \text{span}\{e_0\},$$

the operators $T(t)$ and $T(t)R_\mu$ are uniformly bounded for $t \geq 0$. It is less obvious that on the complementary subspace

$$H_2 = \overline{\text{span}\{e_1, e_2, \ldots\}},$$

the norm of the C_0-semigroup behaves as follows:

$$ct \leq \|T(t)\| \leq Ct, \quad t \geq t_0, \tag{2.5}$$

for some $c, C, t_0 > 0$. Further in this chapter, if two functions $f(t)$ and $g(t)$ meet the relation $cf(t) \leq g(t) \leq Cf(t), \quad t \geq t_0$, it will by denoted by

$$f(t) \sim g(t).$$

In particular (2.5) implies that $\omega_0 = 0$. Also, direct computations (or applying the results from [32]) show that

$$\|T(t)R_\mu\| \leq M, \quad t \geq 0.$$

This means that (2.3) holds despite

$$\{i\} \subset \sigma(A) \cap (i\mathbb{R}) \neq \emptyset.$$

Now we present the main result of this chapter which provides a sufficient condition for (2.3) to hold. This result, in contrast to Theorem 50, allows for the breaking of the condition (2.4).

Theorem 52. *Let $T = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X, with the growth bound $\omega_0 > -\infty$ and the generator A. Suppose $f(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a positive function such that*

$$\limsup_{s\to\infty} \frac{f(t+s)}{f(s)} = e^{\omega_0 t}, \quad t \geq 0, \tag{2.6}$$

$$\|T(t)\| \leq f(t), \quad t \geq 0. \tag{2.7}$$

Assume further that

(a) for any $\lambda \in \sigma(A) \cap (\omega_0 + i\mathbb{R})$ there exists a regular bounded curve $\Gamma_\lambda \subset \mathbb{C}$ enclosing λ, such that $\Gamma_\lambda \cap \sigma(A) = \emptyset$;

(b) for any $\lambda \in \sigma(A) \cap (\omega_0 + i\mathbb{R})$

$$\lim_{t\to\infty} \frac{\|T(t)P_{\Gamma_\lambda}\|}{f(t)} = 0, \tag{2.8}$$

where P_{Γ_λ} is the Riesz projection associated with the curve Γ_λ and the operator A.

Then

$$\lim_{t\to\infty} \frac{\|T(t)R_\mu\|}{f(t)} = 0, \tag{2.9}$$

for any fixed $\mu \in \rho(A)$ (recall that R_μ denotes the operator $R(A,\mu) = (A-\mu)^{-1}$).

Before giving the proof of Theorem 52 we want to state the following remarks:

(a) the idea of using the quotient space defined by the appropriate seminorm was first used in [34] and has been further developed in other papers such as [5] [16] [20] [21] [31];

(b) The work [27] provides a constructive proof of existence of such a function f satisfying (2.6) and (2.7) for an arbitrary C_0-semigroup. The function given in [27] is monotonic and it holds that $f(t_n) = \|T(t_n)\|$ for some unbounded sequence $t_n \in \mathbb{R}_0^+$;

(c) we prove Theorem 52 for the case of $\omega_0 = 0$. For an arbitrary ω_0 one can consider the shifted C_0-semigroup $\{e^{-\omega_0 t}T(t)\}_{t\geq 0}$;

(d) the relation between (2.9) and (2.3) is shown after the proof.

In the proof we will use the construction of the special operator-valued C_0-semigroup introduced in [27]. We note here, that a similar idea has already been used in [20] [21]. Let $\widetilde{X} \subset \mathcal{L}(X)$ be defined as

$$\widetilde{X} = \overline{\{DR_\mu : \, D \in \mathcal{L}(X)\}}, \quad \mu \in \rho(A),$$

where $\overline{Q}$ denotes the closure of the linear hull Q (with respect to the operator norm). Since $\widetilde{X}$ is a closed subspace of the Banach space $\mathcal{L}(X)$, it also is a Banach space. It is clear that $\widetilde{X}$ does not depend on the choice of μ. For the given C_0-semigroup $\{T(t)\}_{t\geq 0}$ on the space X, let us introduce a semigroup of operators on the space $\widetilde{X}$ by:

$$\widetilde{T}(t)\widetilde{B} = \widetilde{B}T(t), \quad \widetilde{B} \in \widetilde{X}, \quad t \geq 0. \tag{2.10}$$

Important properties of this semigroup were shown in [27], namely that $\{\widetilde{T}(t)\}_{t\geq 0}$ forms a C_0-semigroup on $\widetilde{X}$, and that

1. for A and $\widetilde{A}$ being the generators of $\{T(t)\}_{t\geq 0}$ and $\{\widetilde{T}(t)\}_{t\geq 0}$, respectively, it holds that

$$\sigma(\widetilde{A}) \subset \sigma(A); \tag{2.11}$$

2. for $\widetilde{B} \in \widetilde{X}$ and $\mu \in \rho(A)$, it holds that

$$(\widetilde{A} - \mu)^{-1} \widetilde{B} = \widetilde{B}(A - \mu)^{-1}. \tag{2.12}$$

We will also use the following lemma:

Lemma 53. [16] *Let $T = \{T(t)\}_{t \geq 0}$ be a C_0-semigroup of isometries on a Banach space X and denote its generator by A. Then one of the following two cases holds*

(a) $\sigma(A) = \{\mu \in \mathbb{C} : \mathrm{Re}(\mu) \leq 0\}$;

(b) $\sigma(A) \subset (i\mathbb{R})$ and the above C_0-semigroup extends to a strongly continuous group of isometries.

Note that Lemma 53 implies that, for a C_0-semigroup of isometries, if $\partial(\sigma(A)) \neq (i\mathbb{R})$, then $\sigma(A) = \partial(\sigma(A)) \subsetneq (i\mathbb{R})$, where ∂ denotes the boundary of a set. The proof of Theorem 52 is based on the idea used in [31].

Proof of Theorem 52.

Assume that (2.9) does not hold, which means that

$$0 \neq \limsup_{t\to\infty} \frac{\|T(t)R_\mu\|}{f(t)} = \limsup_{t\to\infty} \frac{\|R_\mu T(t)\|}{f(t)} = \limsup_{t\to\infty} \frac{\|\widetilde{T}(t)R_\mu\|}{f(t)}. \tag{2.13}$$

Let us define a following seminorm on $\widetilde{X}$:

$$l(\widetilde{B}) = \limsup_{t\to\infty} \frac{\|\widetilde{T}(t)\widetilde{B}\|}{f(t)}, \quad \widetilde{B} \in \widetilde{X}.$$

It follows from (2.13) that the quotient space $\widetilde{X}/\ker l = \{\widehat{B} = \widetilde{B} + \ker l : \widetilde{B} \in \widetilde{X}\}$ is non-zero. This space can be equipped with a norm different from the natural one ($\|\widehat{B}\|_N := \inf\{\|\widetilde{B}\| : \widetilde{B} \in \widehat{B}\}$) of the following form

$$\|\widehat{B}\|' := l(\widetilde{B}), \quad \widetilde{B} \in \widetilde{X}.$$

Note that, since $\|\widetilde{T}(t)\| \leq \|T(t)\| \leq f(t)$ (see (2.7) and (2.10)), for all $\widetilde{B} \in \widetilde{X}$,

$$l(\widetilde{B}) = \limsup_{t\to\infty} \frac{\|\widetilde{T}(t)\widetilde{B}\|}{f(t)} \leq \|\widetilde{B}\|$$

holds, which means that $\|\widehat{B}\|' \leq \|\widehat{B}\|_N$ and the space $(\widetilde{X}/\ker l, \|\cdot\|')$ may be incomplete. Its completion w.r.t. the norm $\|\cdot\|'$ is denoted by $\widehat{X}$. Let us define the family of operators $\widehat{T}(t), t \geq 0$, by the formula

$$\widehat{T}(t)\widehat{B} = \widetilde{T}(t)\widetilde{B} + \ker l, \quad \widehat{B} \in \widetilde{X}/\ker l \subset \widehat{X}.$$

By applying the property (2.6) for $\omega_0 = 0$, we get

$$\begin{aligned}\|\widehat{T}(t)\widehat{B}\|' &= \limsup_{s\to\infty} \frac{\|\widetilde{T}(t+s)\widetilde{B}\|}{f(t+s)} \frac{f(t+s)}{f(s)} \\ &= \|\widehat{B}\|', \quad \text{ for } \widehat{B} \in \widetilde{X}/\ker l,\end{aligned}$$

Thus, $\{\widehat{T}(t)\}_{t\geq 0}$ is a family of isometries on $\widetilde{X}/\ker l$, w.r.t. the norm $\|\cdot\|'$. It is easy to check that for each $t \geq 0$, $\widehat{T}(t)$ extends to an isometry on $\widehat{X}$ and the family $\{\widehat{T}(t)\}_{t\geq 0}$ is a C_0-semigroup of isometries. Moreover, one can check that

$$\begin{aligned}\widehat{A}\widehat{B} &= \widetilde{A}\widetilde{B} + \ker l, \\ R(\widehat{A}, \mu)\widehat{B} &= R(\widetilde{A}, \mu)\widetilde{B} + \ker l\end{aligned} \tag{2.14}$$

for $\widehat{B} \in \widetilde{X}$, where $\widetilde{A}$ and $\widehat{A}$ are generators of $\{\widetilde{T}(t)\}_{t\geq 0}$ and $\{\widehat{T}(t)\}_{t\geq 0}$, respectively and $R(\widetilde{A}, \mu)$ and $R(\widehat{A}, \mu)$ are the respective resolvent operators at the point μ. It follows from assumption (a) of Theorem 52 and (2.11) that

$$\begin{aligned}(i\mathbb{R}) &\not\subset \sigma(A) \\ (i\mathbb{R}) &\not\subset \sigma(\widetilde{A}).\end{aligned} \tag{2.15}$$

On the other hand, it is shown in [26] [28] that

$$\partial(\sigma(\widehat{A})) \cap (i\mathbb{R}) \subset \sigma(\widetilde{A}) \cap (i\mathbb{R}).$$

This, along with Lemma 53 (b) and (2.15), implies that

$$\partial\sigma(\widehat{A}) = \sigma(\widehat{A}) \subset \sigma(\widetilde{A}) \cap (i\mathbb{R}) \neq (i\mathbb{R}). \tag{2.16}$$

Therefore, again due to Lemma 53, $\{\widehat{T}(t)\}_{t\geq 0}$ extends to a C_0-group of isometries. Now, since $\widehat{A}$ is a generator of a C_0-group of isometries, its spectrum has to be non-empty (see, e.g., [18])

$$\sigma(\widehat{A}) \neq \emptyset.$$

By combining the above with (3.10) and (2.11), we obtain:

$$\emptyset \neq \sigma(\widehat{A}) \subset \sigma(\widetilde{A}) \cap (i\mathbb{R}) \subset \sigma(A) \cap (i\mathbb{R}). \tag{2.17}$$

Note that for the case $\sigma(A) \cap (i\mathbb{R}) = \emptyset$ we obtain here a contradiction. This means that for the case of $\sigma(A) \cap (i\mathbb{R}) = \emptyset$ it holds that

$$\lim_{t\to\infty} \frac{\|T(t)R_\mu\|}{f(t)} = 0.$$

Now assume $\sigma(A) \cap (i\mathbb{R}) \neq \emptyset$. Let us fix λ such that

$$\lambda \in \sigma(\widehat{A}) \subset \sigma(A) \cap (i\mathbb{R}).$$

It follows from the assumption (a) of the Theorem, (2.17), and (2.11) that there exists a bounded curve Γ_λ enclosing λ, such that

$$\Gamma_\lambda \cap \sigma(\widehat{A}) = \Gamma_\lambda \cap \sigma(\widetilde{A}) = \Gamma_\lambda \cap \sigma(A) = \emptyset.$$

Let $\widetilde{P}_{\Gamma_\lambda}$ and $\widehat{P}_{\Gamma_\lambda}$ be the Riesz projections in $\widetilde{X}$ and $\widehat{X}$, respectively, corresponding to the curve Γ_λ. One can see from (2.14), that for $\widehat{B} \in \widetilde{X}/\ker l$

$$\widehat{P}_{\Gamma_\lambda}\widehat{B} = \widetilde{P}_{\Gamma_\lambda}\widetilde{B} + \ker l. \tag{2.18}$$

Furthermore, the projections $\widetilde{P}_{\Gamma_\lambda}$ and $\widehat{P}_{\Gamma_\lambda}$ split the spaces $\widetilde{X}$ and $\widehat{X}$ into direct sums $\widetilde{Z}_1 \oplus \widetilde{Z}_2$ and $\widehat{Z}_1 \oplus \widehat{Z}_2$, respectively (see Definition 16), so that

$$\begin{aligned}
\widetilde{Z}_1 &:= \widetilde{P}_{\Gamma_\lambda}\widetilde{X},\\
\widetilde{Z}_2 &:= (I - \widetilde{P}_{\Gamma_\lambda})\widetilde{X},\\
\widehat{Z}_1 &:= \widehat{P}_{\Gamma_\lambda}\widehat{X},\\
\widehat{Z}_2 &:= (I - \widehat{P}_{\Gamma_\lambda})\widehat{X}.
\end{aligned}$$

Clearly the spectra of the restricted operators $\widetilde{A}|_{\widetilde{Z}_1}$ and $\widetilde{A}|_{\widetilde{Z}_2}$ are intersections of $\sigma(\widetilde{A})$ with regions inside and outside Γ_λ, respectively, with an analogous property for the operator $\widehat{A}$ (see Definition 16). Now, since the set $\sigma(\widehat{A})$ is a boundary set, it consists only of approximate eigenvalues (see Lemma 4). This means that for the chosen λ there exists a sequence $\{\widehat{B}_k : \|\widehat{B}_k\|' = 1\}$ such that

$$\|\widehat{A}\widehat{B}_k - \lambda\widehat{B}_k\|' \to 0 \text{ as } k \to \infty. \tag{2.19}$$

Now, $\{\widehat{B}_k\}$ can be split into sequences

$$\widehat{B}_k = \widehat{B}_k^{(1)} + \widehat{B}_k^{(2)},$$

where

$$\widehat{B}_k^{(1)} \in \widehat{Z}_1, \quad \widehat{B}_k^{(2)} \in \widehat{Z}_2.$$

Then it follows from (2.19), that

$$\begin{aligned}
\|\widehat{A}\widehat{B}_k^{(1)} - \lambda\widehat{B}_k^{(1)}\|' &\to 0,\\
\|\widehat{A}\widehat{B}_k^{(2)} - \lambda\widehat{B}_k^{(2)}\|' &\to 0,
\end{aligned}$$

as $k \to \infty$. Subsequently,

$$\|\widehat{B}_k^{(2)}\| \to 0,$$

since otherwise λ would belong to $\sigma(\widehat{A}|_{\widehat{Z}_2})$, giving a contradiction. In consequence

$$\|\widehat{B}_k^{(1)}\|' \geq \frac{1}{2}$$

for k large enough. Furthermore, by the density of $\widetilde{X}/\ker l$ in $\widehat{X}$ and by the boundedness of $\widehat{A}|_{Z_1}$, the sequence $\widehat{B}_k^{(1)}$ can be chosen from $\widehat{P}_{\Gamma_\lambda}(\widetilde{X}/\ker l) \subset \widehat{Z}_1$. Subsequently, from (2.18), we get

$$\widehat{B}_k^{(1)} = \widehat{P}_{\Gamma_\lambda}\widehat{B}_k = \widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k + \ker l,$$

for some sequence $\widetilde{B}_k \in \widetilde{X}$. Then the following estimate holds for large enough k

$$\frac{1}{2} \leq \|\widehat{B}_k^{(1)}\|' = \|\widehat{P}_{\Gamma_\lambda}\widehat{B}_k\|' = \|\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k + \ker l\|' = l(\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k) = \limsup_{t\to\infty}\frac{\|\widetilde{T}(t)\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k\|}{f(t)}, \tag{2.20}$$

Now, by integrating the equation (2.12), we obtain

$$\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k = \oint_{\Gamma_\lambda}(\widetilde{A}-\mu)^{-1}\widetilde{B}_k \mathrm{d}\mu = \oint_{\Gamma_\lambda}\widetilde{B}_k(A-\mu)^{-1}\mathrm{d}\mu = \widetilde{B}_k P_{\Gamma_\lambda}, \tag{2.21}$$

where we have used the analicity of the resolvent operator function and the boundedness of $\widetilde{B}_k$ as an operator from $\mathcal{L}(X)$ to $\mathcal{L}(X)$ (treated as a multiplication operator). Using (2.21), and the definition of $\widetilde{T}(t)\widetilde{B} = \widetilde{B}T(t)$ in (2.20), we get

$$\begin{aligned}\frac{1}{2} &\leq \limsup_{t\to\infty}\frac{\|\widetilde{T}(t)\widetilde{P}_{\Gamma_\lambda}\widetilde{B}_k\|}{f(t)} = \limsup_{t\to\infty}\frac{\|\widetilde{B}_k P_{\Gamma_\lambda}T(t)\|}{f(t)}\\ &\leq \limsup_{t\to\infty}\frac{\|\widetilde{B}_k\|\|P_{\Gamma_\lambda}T(t)\|}{f(t)} = 0,\end{aligned}$$

where to evaluate the limit we have used the assumption (2.8). This yields a contradiction, thus (2.13) cannot hold, i.e.,

$$\lim_{t\to\infty}\frac{\|T(t)R_\mu\|}{f(t)} = 0.$$

□

Remark 54. *For bounded C_0-semigroups ($\|T(t)\| \leq M$ for $t \geq 0$) with $\sigma(A) \cap (i\mathbb{R}) = \emptyset$, by taking $f \equiv M$ one can easily see that Theorem 52 implies the sufficiency part of Theorem 48.*

Remark 55. *In the assertion of Theorem 52, one can replace the function $f(t)$ satisfying the conditions of the Theorem with $\|T(t)\|$ whenever*

$$\|T(t)\| \sim f(t).$$

Examples of such semigroups are given in the next section.

2.3 Application to generators with a countable purely imaginary simple spectrum

Now we will provide some examples of application of Theorem 52 to unbounded semigroups in Hilbert spaces. The generators of these C_0-semigroups have a countable purely imaginary simple spectrum such that the eigenvectors form a linearly dense set. Due to the [37] [39], we know that the eigenvalues cannot be uniformly separated (i.e., $\inf\{|\lambda_k - \lambda_m| : k, m \in \mathbb{N}, k \neq m\} > 0$) since, for this being the case, the eigenvectors would form a Riesz basis, which in turn would imply the boundedness of the semigroup and thus, due to Theorem 48, (2.3) could not hold (since $\sigma(A) \cap (i\mathbb{R}) \neq \emptyset$). Therefore the following examples are unbounded C_0-semigroups (for which the eigenvalues are not uniformly separated). We begin with a rather simple example, namely

Example 56. *Let $\{e_n\}_{n=1}^{\infty}$ be the orthonormal basis of a Hilbert space H. Define the operator $A : D(A) \subset H \to H$ as follows:*

$$A|_{H_n} := A_n := \begin{bmatrix} ni + \frac{i}{n} & 1 \\ 0 & ni - \frac{i}{n} \end{bmatrix},$$

where

$$H_n = \operatorname{span}\{e_{2n-3}, e_{2n-2}\},\ n = 2, 3, 4 \ldots$$

For each $n \geq 2$ consider the curve Γ_n enclosing the pair of eigenvalues $\{(ni + \frac{i}{n}), (ni - \frac{i}{n})\}$. Then the image of the Riesz projection corresponding to the curve Γ_n is equal H_n. One can directly check that

$$e^{A_n t} := T_n(t) = e^{tni} \begin{bmatrix} e^{i\frac{t}{n}} & n \sin \frac{t}{n} \\ 0 & e^{-i\frac{t}{n}} \end{bmatrix},$$

Since $\|T(t)\| = \sup\limits_{n\geq 2} \|T_n(t)\|$, we have

$$\|T(t)\| \sim t.$$

It is easy to see, that $f(t) := t$ has the desired properties (2.6), (2.7) up to the multiplication by a constant. Clearly assumptions (a) and (b) of Theorem 52 are satisfied. Therefore (2.3) holds, i.e.,

$$\frac{\|T(t)A^{-1}\|}{t} \to 0, \quad t \to \infty. \tag{2.22}$$

Moreover, for this simple case, we can calculate the decay rate of (2.22), namely

$$T_n(t)A_n^{-1} = \frac{in}{1-n^4} e^{tni} \begin{bmatrix} (n^2-1)e^{i\frac{t}{n}} & (n^2-1)n \sin \frac{t}{n} + ine^{-i\frac{t}{n}} \\ 0 & (n^2+1)e^{-i\frac{t}{n}} \end{bmatrix},$$

hence

$$\|T(t)A^{-1}\| = \sup_{n\geq 2} \|T_n(t)A_n^{-1}\| \sim 1, \quad t \geq 0.$$

Finally, it follows that

$$\frac{\|T(t)A^{-1}\|}{\|T(t)\|} \sim \frac{1}{t} \to 0, \quad t \to \infty.$$

Now we will give an example of a family of unbounded C_0-semigroups that have a simple countable purely imaginary spectrum and the eigenvectors are linearly dense but do not form a Riesz basis. This family was described in [29] [30]. The elements of this family are constructed as follows. Let $(H, \|\cdot\|)$ be a Hilbert space with the orthonormal basis $\{e_n\}_{n=2}^{\infty}$. For the sequence

$$\lambda_n = i \log n,\ n = 2, 3, \ldots$$

define the C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ by

$$T(t)e_n = e^{t\lambda_n} e_n,$$

For a given $N \in \mathbb{N} \setminus \{0\}$ we are able to choose a new norm $\|\cdot\|_N$ on H, dominated by $\|\cdot\|$ such that:

(a) The C_0-semigroup T naturally extends to a C_0-semigroup $\widetilde{T}$ on the completion of $(H, \|\cdot\|_N)$, say $\widetilde{H}_N$;

(b) there exist constants $m, M > 0$ such that

$$mt^N \leq \|\widetilde{T}(t)\| \leq Mt^N + 1, \quad t \geq 0. \tag{2.23}$$

See [29] [30] for a detailed construction and estimations. Denote the generator of $\widetilde{T}$ by $\widetilde{A}$. It is shown in [29] that

$$\sigma(\widetilde{A}) = \sigma_P(\widetilde{A}) = \bigcup_{n\geq 2} \{i \log n\}.$$

We are going to show that the C_0-semigroup $\widetilde{T}$ meets the assumptions of Theorem 52, however first we will compute $\frac{\|\widetilde{T}(t)\widetilde{A}^{-1}\|}{\|\widetilde{T}(t)\|}$ "by hand" for the case of $N = 1$. Before we do that we should show some basic properties of the space $\widetilde{H}_N$, as shown in [29] [30]. Consider the backward difference operator

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The space $\widetilde{H}_N$ is defined as the completion of

$$\Big\{x = (\mathfrak{f})\sum_{n=2}^{\infty} c_n e_n : \{c_n\}_{n=1}^{\infty} \in l_2(\Delta^N)\Big\},$$

with respect to the norm on this space defined as:

$$\|x\|_N = \Big\|(\mathfrak{f})\sum_{n=2}^{\infty} c_n e_n\Big\|_N = \Big\|\sum_{n=2}^{\infty}\sum_{j=0}^{N}(-1)^j C_N^j c_{n-j} e_n\Big\|, \tag{2.24}$$

where $l_2(\Delta^N) = \big\{x = \{c_n\}_{n=2}^{\infty}, c_n \in \mathbb{C} : \Delta^N x \in l_2\big\}$ and $(\mathfrak{f})$ denotes the formal series. The norm without a subscript denotes the norm in the initial Hilbert space H, and C_N^j denote the binomial coefficients $\binom{N}{j}$. The action of the generator, resolvent at the point 0 and product of the C_0-semigroup and the resolvent are as follows:

$$\begin{aligned} \widetilde{A}e_n &= i\log(n)e_n, \quad n \geq 2, \\ \widetilde{A}^{-1}e_n &= \frac{1}{i\log(n)}e_n \quad n \geq 2, \\ \widetilde{T}(t)\widetilde{A}^{-1}e_n &= \frac{e^{it\log(n)}}{i\log(n)}e_n, \quad n \geq 2, \\ \widetilde{T}(t)\widetilde{A}^{-1}x &= \sum_{n=2}^{\infty} c_n \frac{e^{it\log(n)}}{i\log(n)}e_n. \end{aligned}$$

Let us consider the simplest case of $\widetilde{T}$ when $N = 1$.

Example 57. *Consider* $\widetilde{T} : \widetilde{H}_1 \to \widetilde{H}_1$, *then*

$$\|\widetilde{x}\|_1 = \Big(\sum_{n=2}^{\infty} \Big| c_{n+1} - c_n \Big|^2 + |c_2|^2 \Big)^{\frac{1}{2}}, \quad \widetilde{x} \in \widetilde{H}_1.$$

We will prove that for this case

$$\frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{\|\widetilde{T}(t)\|} \sim \frac{1}{\log(t)}. \tag{2.25}$$

In further considerations we will use the following inequality for the sequence $\{c_n\}_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty} \frac{|c_n|^2}{n^2} \leq 4 \sum_{n=1}^{\infty} |c_{n+1} - c_n|^2, \quad c_n \in \mathbb{C}, \tag{2.26}$$

which is a special case (see [29] [30]*) of the Hardy's inequality which holds for any non-negative sequence of real numbers* $\{a_n\}_{n=1}^{\infty}$*:*

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=1}^{n} a_k \Big)^p \leq \Big(\frac{p}{p-1} \Big)^p \sum_{n=1}^{\infty} a_n^p, \quad a_n \geq 0$$

for $p = 2$*. To prove* (2.25) *we will first estimate* $\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\|_1^2$ *by*

$$\begin{aligned}
\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\|_1^2 &= \sum_{n=2}^{\infty} \Big| c_{n+1} \frac{e^{it\log(n+1)}}{i\log(n+1)} - c_n \frac{e^{it\log(n)}}{i\log(n)} \Big|^2 + |c_2|^2 \\
&\leq 2 \sum_{n=2}^{\infty} \Big| c_{n+1} \frac{e^{it\log(n+1)}}{i\log(n+1)} - c_{n+1} \frac{e^{it\log(n)}}{i\log(n)} \Big|^2 + 2 \sum_{n=2}^{\infty} \Big| (c_{n+1} - c_n) \frac{e^{it\log(n)}}{i\log(n)} \Big|^2 + |c_2|^2.
\end{aligned}$$

The second and third elements of the r.h.s of the above inequality are clearly bounded by $B\Big(\frac{t}{\log(t)}\Big)^2 \|\widetilde{x}\|_1^2$ *and* $C\Big(\frac{t}{\log(t)}\Big)^2 \|\widetilde{x}\|_1^2$, $B, C > 0$ *for* $t > e$*. We only need to look at the first sum then.*

$$\begin{aligned}
&\sum_{n=2}^{\infty} \Big| c_{n+1} \frac{e^{it\log(n+1)}}{i\log(n+1)} - c_{n+1} \frac{e^{it\log(n)}}{i\log(n)} \Big|^2 \\
&= \sum_{n=2}^{\infty} \Big| \frac{c_{n+1}}{n} \frac{n(e^{it\log(n)}\log(n+1) - e^{it\log(n)}\log(n))}{\log(n+1)\log(n)} + \frac{c_{n+1}}{n} \frac{n(e^{it\log(n)}\log(n) - e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)} \Big|^2 \\
&\leq 2 \sum_{n=2}^{\infty} \Big| \frac{c_{n+1}}{n} \frac{n(e^{it\log(n)}\log(n+1) - e^{it\log(n)}\log(n))}{\log(n+1)\log(n)} \Big|^2 + \Big| \frac{c_{n+1}}{n} \frac{n(e^{it\log(n)}\log(n) - e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)} \Big|^2 \\
&\leq 2 \sum_{n=2}^{\infty} \Big| \frac{c_{n+1}}{n} \frac{n\log(1+\frac{1}{n})}{\log(n+1)\log(n)} \Big|^2 + 2 \sum_{n=2}^{\infty} \Big| \frac{c_{n+1}}{n} \frac{n(1 - e^{it\log(1+\frac{1}{n})})}{\log(n+1)} \Big|^2.
\end{aligned}$$

The first of the above sums, due to Hardy's inequality (see (2.26)*), is bounded by* $D\|\widetilde{x}\|_1^2$*, and thus by*

$D\Big(\frac{t}{\log(t)}\Big)^2\|\widetilde{x}\|_1^2$ *for* $t>e$. *We estimate the remaining sum by splitting it into two* t*-dependent sums.*

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\Big|\frac{c_{n+1}}{n}\frac{n(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)}\Big|^2\\
=&\sum_{2\le n<t}\Big|\frac{c_{n+1}}{n}\frac{n(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)}\Big|^2+\sum_{n\ge t}\Big|\frac{c_{n+1}}{n}\frac{n(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)}\Big|^2\\
\le&E\sum_{2\le n<t}\Big|\frac{c_{n+1}}{n}\Big|^2\Big(\frac{t}{\log(t)}\Big)^2+\sum_{n\ge t}\Big|\frac{c_{n+1}}{n}\frac{tn\log(1+\frac{1}{n})(1-e^{it\log(1+\frac{1}{n})})}{\log(n+1)t\log(1+\frac{1}{n})}\Big|^2\\
\le&E\sum_{2\le n<t}\Big|\frac{c_{n+1}}{n}\Big|^2\Big(\frac{t}{\log(t)}\Big)^2+F\sum_{n\ge t}\Big|\frac{c_{n+1}}{n}\frac{(1-e^{it\log(1+\frac{1}{n})})}{t\log(1+\frac{1}{n})}\Big|^2\Big(\frac{t}{\log(t)}\Big)^2\\
\le&(E+G)\sum_{n=2}^{\infty}\Big|\frac{c_{n+1}}{n}\Big|^2\Big(\frac{t}{\log(t)}\Big)^2.
\end{aligned}
$$

Where we have used the boundedness of $s\log(1+\frac{1}{s})$ *and* $\frac{1-e^{is}}{s}$ *for* $s\in\mathbb{R}^+$. *Again due to* (2.26), *we obtain*

$$
\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\|_1\le(B+C+D+4E+4G)^{\frac{1}{2}}\frac{t}{\log(t)}\|\widetilde{x}\|_1.
$$

Thus

$$
\|\widetilde{T}(t)\widetilde{A}^{-1}\|\le M_0\frac{t}{\log(t)},\tag{2.27}
$$

for some $M_0>0$ *and* $t>e$. *We will now prove the opposite inequality*

$$
m_0\frac{t}{\log(t)}\le\|\widetilde{T}(t)\widetilde{A}^{-1}\|\tag{2.28}
$$

for some $m_0>0$. *First, we observe that due to the reverse triangle inequality, it holds that*

$$
\begin{aligned}
&\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\|_1=\Big(\sum_{n=2}^{\infty}\Big|c_{n+1}\frac{e^{it\log(n+1)}}{i\log(n+1)}-c_n\frac{e^{it\log(n)}}{i\log(n)}\Big|^2+|c_2|^2\Big)^{\frac{1}{2}}\\
&\ge\Big(\sum_{n=2}^{\infty}\Big|c_{n+1}\frac{(e^{it\log(n)}\log(n)-e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)}\Big|^2\Big)^{\frac{1}{2}}\\
&-\Big(\sum_{n=2}^{\infty}\Big|c_{n+1}\frac{e^{it\log(n+1)}}{i\log(n+1)}-c_{n+1}\frac{e^{it\log(n)}}{i\log(n)}\Big|^2\Big)^{\frac{1}{2}}\\
&-\Big(\sum_{n=2}^{\infty}\Big|(c_{n+1}-c_n)\frac{e^{it\log(n)}}{i\log(n)}\Big|^2\Big)^{\frac{1}{2}}-|c_2|\qquad.
\end{aligned}
$$

It follows from previous considerations that

$$
\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}\|_1\ge\Big(\sum_{n=2}^{\infty}\Big|c_{n+1}\frac{(e^{it\log(n)}\log(n)-e^{it\log(n+1)}\log(n))}{\log(n+1)\log(n)}\Big|^2\Big)^{\frac{1}{2}}-C\|\widetilde{x}\|_1
$$

for some $C>0$. *Thus, in order to prove* (2.28), *it suffices to show that*

$$
\sum_{n=2}^{\infty}\Big|c_{n+1}\frac{(e^{it\log(n)}-e^{it\log(n+1)})}{\log(n+1)}\Big|^2\ge m_1^2\Big(\frac{t}{\log(t)}\Big)^2\|\widetilde{x}\|_1^2\tag{2.29}
$$

for some $m_1 > 0$ and $t > e$. To this end, we construct for each $t > e$ an element in $\widetilde{H}_1$ in the following way

$$\widetilde{x}^{(t)} = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n^{(t)} e_n, \quad j \in \mathbb{N}, \quad \text{where}$$

$$c_n^{(t)} = \begin{cases} n, & \text{if } n \leq 2t, \\ 4t - n, & \text{if } 2t < n \leq 4t, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$\|\widetilde{x}^{(t)}\|_1^2 \leq 4t. \tag{2.30}$$

Now, the following estimate holds (see (2.29))

$$\begin{aligned}
&\sum_{n=2}^{\infty} \Big| c_{n+1}^{(t)} \frac{(e^{it\log(n)} - e^{it\log(n+1)})}{\log(n+1)} \Big|^2 \\
&\geq \sum_{t \leq n \leq 2t} \Big| t \frac{1 - e^{it\log(1+\frac{1}{n})})}{\log(n+1)} \Big|^2 \\
&\geq \Big(\frac{t}{\log(4t)}\Big)^2 \sum_{t \leq n \leq 2t} \Big| \frac{1 - e^{it\log(1+\frac{1}{n})})}{it\log(1+\frac{1}{n})} it\log(1+\frac{1}{n}) \Big|^2 \\
&\geq \Big(\frac{t}{\log(4t)}\Big)^2 \sum_{t \leq n \leq 2t} \Big| \frac{1 - e^{it\log(1+\frac{1}{n})})}{t\log(1+\frac{1}{n})} \log(1+\frac{1}{2t})^t \Big|^2 \\
&\geq \Big(\frac{Ct}{\log(4t)}\Big)^2 \sum_{0 \leq n \leq t} \Big| \frac{1 - e^{it\log(1+\frac{1}{n+t})})}{t\log(1+\frac{1}{n+t})} \Big|^2 \\
&\geq \Big(\frac{Ct^2}{\log(4t)}\Big)^2 \sum_{0 \leq n \leq t} D \\
&\geq \Big(\frac{Ct}{\log(4t)}\Big)^2 \frac{t}{2} D
\end{aligned}$$

for $t > e$ and some $C, D > 0$ independent of $t > e$. Combining the above with (2.29) and (2.30) gives

$$m_0 \frac{t}{\log(t)} \leq \frac{\|\widetilde{T}(t)\widetilde{A}^{-1}\widetilde{x}^{(t)}\|_1}{\|\widetilde{x}^{(t)}\|_1}$$

for $t > e$. Together with (2.27) this shows that

$$m_0 \frac{t}{\log(t)} \leq \|\widetilde{T}(t)\widetilde{A}^{-1}\| \leq M_0 \frac{t}{\log(t)}$$

for $t > e$. This implies, due to (2.23) that

$$m_0' \frac{1}{\log(t)} \leq \frac{\|\widetilde{T}(t)\widetilde{A}^{-1}\|}{\|\widetilde{T}(t)\|} \leq M_0' \frac{1}{\log(t)} \tag{2.31}$$

for some $m_0', M_0' > 0$ *and* $t > e$ *or, equivalently,*

$$\frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{\|\widetilde{T}(t)\|} \sim \frac{1}{\log(t)}$$

for $t > e$ *and arbitrary* $\mu \in \rho(\widetilde{A})$*. Thus*

$$\frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{\|\widetilde{T}(t)\|} \to 0, \qquad \text{as } t \to \infty.$$

□

A similar result can be obtained with the use of Theorem 52. We are going to check that the C_0*-semigroup* $\widetilde{T}$ *meets the assumptions of Theorem 52 for arbitrary* $N \in \mathbb{N} \setminus \{0\}$*. Indeed, for each* $\lambda_n = i \log n$ *one can choose* Γ_n *surrounding only one point of* $\sigma(\widetilde{A})$*, namely* λ_n*. Note also that, for* $x \in H \subset \widetilde{H}$*,*

$$\widetilde{A}x = Ax,$$
$$R(\widetilde{A}, \lambda)x = R(A, \lambda)x,$$
$$\widetilde{P}_{\Gamma_n}x = P_{\Gamma_n}x.$$

Hence, due to density of H *in* $\widetilde{H}$

$$\widetilde{T}(t)\widetilde{P}_{\Gamma_n}\widetilde{x} = e^{it\log n}\widetilde{P}_{\Gamma_n}\widetilde{x}, \quad \widetilde{x} \in \widetilde{H}.$$

It is easy to see that the function $f(t) \equiv Mt^N + 1$ *has the properties (2.6), (2.7), and that the following holds:*

$$\frac{\|\widetilde{T}(t)\widetilde{P}_{\Gamma_n}\|}{f(t)} \leq \frac{\|\widetilde{P}_{\Gamma_n}\|}{f(t)} \leq \frac{\|\widetilde{P}_{\Gamma_n}\|}{Mt^N} \to 0, \qquad \text{as } t \to \infty, \quad n > 0.$$

This means that the C_0*-semigroup meets the assumption (b) of Theorem 52. Application of the presented result yields*

$$0 = \lim_{t\to\infty} \frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{Mt^N + 1} = \lim_{t\to\infty} \frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{t^N} = \lim_{t\to\infty} \frac{\|\widetilde{T}(t)\widetilde{R}_\mu\|}{\|\widetilde{T}(t)\|},$$

for any fixed $\mu \in \rho(\widetilde{A})$*.*

□

The application of Theorem 52 rendered much shorter calculations for arbitrary N than calculations "by hand" for the simplest case of $N = 1$ (even though only the calculation which show that $\frac{\|\widetilde{T}(t)\widetilde{A}^{-1}\|}{\|\widetilde{T}(t)\|} \leq M_0' \frac{1}{\log(t)}$ are relevant in this comparison). One can only expect the calculations to become more complicated for larger N.

Chapter 3

Delay differential equations of the neutral type in infinite-dimensional separable Hilbert spaces

3.1 Introduction

Consider the following delay differential equation of the neutral type in an arbitrary Banach space X:

$$\dot{z}(t) = A\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\mathrm{d}\theta, \tag{3.1}$$

with the initial condition vector function $z_0(\cdot)$ belonging to the Sobolev space $W^{1,p}([-1,0];X)$ (cf. Definition 44) for some fixed $p \geq 1$, and where A is a bounded operator on X and $A_{2,3}(\cdot)$ are strongly measurable operator-valued functions (see Definition 34) belonging to the space $L^q([-1,0];\mathcal{L}(X))$, i.e., such that

$$\int_{-1}^{0} \|A_{2,3}(\theta)\|^q_{\mathcal{L}(X)}\, \mathrm{d}\theta < \infty, \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1. \tag{3.2}$$

It this chapter we represent the equation (3.1), as first introduced by Burns et al. [7] for the finite-dimensional case, in a product space as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix}, \quad \mathcal{A}\begin{pmatrix} y \\ z(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{z}(\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(\theta)\mathrm{d}\theta \\ \mathrm{d}z(\theta)/\mathrm{d}\theta \end{pmatrix} \tag{3.3}$$

where $z_t(\cdot) = z(t+\cdot)$ and the domain of the operator $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{(y, z(\cdot)) : z(\cdot) \in W^{1,p}([-1,0];X), y = z(0) - Az(-1)\} \subset X \times L^p([-1,0];X).$$

For the space X equal $\mathbb{C}^n$ the system (3.3) has been thoroughly analyzed in terms of spectral analysis, stability and stabilizability in [22–25]. A key tool for stability analysis in the mentioned works is the existence of a Riesz basis of $\mathcal{A}$-invariant subspaces constructed from the Riesz projections of the operator $\mathcal{A}$. As it turns out, such basis of subspaces exists for the more general case of $X = H$, where H is an arbitrary separable Hilbert space for a certain class of perturbation integral operators in (3.1), which is the main result of this chapter (Theorem 74). We note here that, in contrast to the case of $X = \mathbb{C}^n$, for the infinite-dimensional case these subspaces are infinite-dimensional as well. We also note here that, in contrast to [25], we do not analyze the nature of the spectrum, however the infinite-dimensional case of (3.1) allows for the spectrum of $\mathcal{A}$ to be an uncountable set (see Corollary 62), while for the case of $X = \mathbb{C}^n$ such case cannot occur.

3.2 Preliminary results

This section contains some preliminary results, including the proof of the generation of a C_0-semigroup by the operator $\mathcal{A}$. The first lemma, combined with some classical results concerning the characterization of C_0-semigroup generators (Theorem 14) will give means to prove that the operator $\mathcal{A}$ representing the system (3.1) via the equation (3.3) generates a C_0-semigroup whenever the system satisfies the condition (3.2).

Lemma 58. *For any initial state*

$$\begin{pmatrix} y \\ z_0(\cdot) \end{pmatrix} \in D(\mathcal{A}) \subset X \times L^p([-1,0];X),$$

there exists a unique classical solution of (3.3) *whenever the system satisfies the condition* (3.2) *for an arbitrary Banach space* X*.*

Proof. The idea of the proof is similar as for the case when the space X is equal $\mathbb{C}^n$, first formulated by *R. Rabah, G.M. Sklyar* and *A.V. Rezounenko*[1]. To those authors we owe the form of the operator B and the space $Y(\beta)$, and the idea of the use of the Banach Fixed Point Theorem. The extension to an arbitrary Banach space with the system (3.1) satisfying the condition (3.2) is due to this work. First we set an arbitrary initial state

$$\begin{pmatrix} y \\ z_0(\cdot) \end{pmatrix} \in D(\mathcal{A})$$

and define the function $\hat{z} \in W^{1,p}([-1,\beta];X)$ in the following way:

$$\hat{z}(t) \equiv \begin{cases} z_0(t), & \text{for} \quad t \in [-1,0], \\ z_0(0), & \text{for} \quad t \in [0,\beta], \end{cases} \tag{3.4}$$

where the parameter $\beta > 0$, as for now arbitrary, will be set later. Now consider the function

$$W^{1,p}([-1,\beta];X) \ni z(t) = \hat{z}(t) + \xi(t)$$

where $\xi \in W^{1,p}([-1,\beta];X)$ and $\xi(t) \equiv 0$ for $t \in [-1,0]$. Using the Banach Fixed Point Theorem we will find $\xi(t)$ in such a way, that $z(t)$ will satisfy (3.3) for $t \in [-1,\beta]$, for some $\beta > 0$ with $z(t) \equiv z_0(t)$

[1] This result cannot be found on-line.

for $t \in [-1, 0]$. Let us now define the closed subspace $Y(\beta)$ of $W^{1,p}([-1, \beta]; X)$ as

$$Y(\beta) \equiv \left\{ \xi(\cdot) \in W^{1,p}([-1, \beta]; X) : \xi(t) \equiv 0 \quad \text{for} \quad t \in [-1, 0] \right\}$$

Note that $Y(\beta)$ is a Banach space. Consider the operation $B : Y(\beta) \to Y(\beta)$

$$B(\xi)(t) \equiv \begin{cases} I(t) + \int_{-1}^{0} A_2(\theta)\xi(t+\theta)\mathrm{d}\theta + \int_0^t \left\{ \int_{-1}^{0} A_3(\theta)\xi(\tau+\theta)\mathrm{d}\theta \right\} \mathrm{d}\tau, & \text{for } t \in [0, \beta], \\ 0, \quad \text{for } t \in [-1, 0]. \end{cases} \tag{3.5}$$

where

$$\begin{aligned} I(t) = A\left[z_0(t-1) - z_0(-1)\right] + \int_{-1}^{0} A_2(\theta)\hat{z}(t+\theta)\mathrm{d}\theta \\ - \int_{-1}^{0} A_2(\theta)z_0(\theta)\mathrm{d}\theta + \int_0^t \left\{ \int_{-1}^{0} A_3(\theta)\hat{z}(\tau+\theta)\mathrm{d}\theta \right\} \mathrm{d}\tau. \end{aligned}$$

Note that the vector function $I(t)$ depends only on the initial data. The fixed point of the operator B yields a solution of $z(t) = \hat{z}(t) + \xi(t)$, which we are looking for. This can be seen from the following considerations: Let $\xi(t) \in Y(\beta)$ be a fixed point of the operator $B : Y(\beta) \to Y(\beta)$. Then it holds that

$$\xi(t) = \begin{cases} I(t) + \int_{-1}^{0} A_2(\theta)\xi(t+\theta)\mathrm{d}\theta + \int_0^t \left\{ \int_{-1}^{0} A_3(\theta)\xi(\tau+\theta)\mathrm{d}\theta \right\} \mathrm{d}\tau, & \text{for } t \in [0, \beta], \\ 0, \quad \text{for } t \in [-1, 0]. \end{cases}$$

which, after differentiating, gives

$$\dot{\xi}(t) = \begin{cases} A\dot{z}_0(t-1) + \int_{-1}^{0} A_2(\theta)\dot{\hat{z}}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)\hat{z}(t+\theta)\mathrm{d}\theta \\ + \int_{-1}^{0} A_2(\theta)\dot{\xi}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)\xi(t+\theta)\mathrm{d}\theta, \text{ for } t \in [0, \beta], \\ 0, \quad \text{for } t \in [-1, 0]. \end{cases}$$

Due to the form of $\hat{z}(t)$ (see (3.4)), keeping in mind that $\dot{z}_0(t-1) \equiv \dot{\hat{z}}(t-1)$ for $t \in [0, \beta]$ and $\xi(t-1) \equiv \dot{\xi}(t-1) \equiv 0$ for $t \in [0, \beta]$, for $\beta < 1$, we get

$$\dot{\hat{z}}(t) + \dot{\xi}(t) = \begin{cases} A(\dot{\hat{z}}(t-1) + \dot{\xi}(t-1)) + \int_{-1}^{0} A_2(\theta)(\dot{\hat{z}} + \dot{\xi})(t+\theta)\mathrm{d}\theta \\ + \int_{-1}^{0} A_3(\theta)(\hat{z} + \xi)(t+\theta)\mathrm{d}\theta, \text{ for } t \in [0, \beta], \\ \dot{z}_0(t), \quad \text{for } t \in [-1, 0]. \end{cases}$$

Thus $z(t) = \hat{z}(t) + \xi(t)$ is a solution to (3.3) on $[0, \beta]$ satisfying the initial condition $z_0(t)$ for $t \in [-1, 0]$ whenever $\xi(t)$ is a fixed point of the operator B.

It remains to find β such that the operator B defined on the space $Y(\beta)$ is a contraction mapping, i.e.,

$$\|B\xi_1(\cdot) - B\xi_2(\cdot)\| \leq L\|\xi_1(\cdot) - \xi_2(\cdot)\|, \quad L < 1,$$

and the existence of the (unique) fixed point will follow from the Banach Fixed Point Theorem. Keeping in mind that $\xi_1(t) - \xi_2(t) \equiv 0$ for $t \in [-1, 0]$, and using the fact that the norm of a Bochner integral is less or equal than the integral of the norm of the integrand (see Proposition 31), we proceed by writing

$$
\begin{aligned}
&\|B(\xi_1)(\cdot) - B(\xi_2)(\cdot)\|_{W^{1,p}([-1,\beta];X)} \\
&= \left\| \int_{-1}^{0} A_2(\theta)\,(\xi_1 - \xi_2)\,(\cdot + \theta)\mathrm{d}\theta + \int_0^{(\cdot)} \left\{ \int_{-1}^{0} A_3(\theta)(\xi_1 - \xi_2)(\tau + \theta)\mathrm{d}\theta \right\} \mathrm{d}\tau \right\|_{W^{1,p}([-1,\beta];X)} \\
&= \left(\int_0^{\beta} \left\| \int_{-1}^{0} A_2(\theta)(\xi_1 - \xi_2)(t + \theta)\mathrm{d}\theta + \int_0^{t} \left\{ \int_{-1}^{0} A_3(\theta)(\xi_1 - \xi_2)(\tau + \theta)\mathrm{d}\theta \right\} \mathrm{d}\tau \right\|_X^p \mathrm{d}t \right. \\
&\left. + \int_0^{\beta} \left\| \int_{-1}^{0} A_2(\theta)(\dot{\xi}_1 - \dot{\xi}_2)(t + \theta)\mathrm{d}\theta + \left\{ \int_{-1}^{0} A_3(\theta)(\xi_1 - \xi_2)(t + \theta)\mathrm{d}\theta \right\} \right\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&\le \left(\int_0^{\beta} \left\| \int_{-1}^{0} A_2(\theta)(\xi_1 - \xi_2)(t + \theta)\mathrm{d}\theta \right\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} + \left(\int_0^{\beta} \left\| \int_0^{t} \left\{ \int_{-1}^{0} A_3(\theta)(\xi_1 - \xi_2)(\tau + \theta)\mathrm{d}\theta \right\} \mathrm{d}\tau \right\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&+ \left(\int_0^{\beta} \left\| \int_{-1}^{0} A_2(\theta)(\dot{\xi}_1 - \dot{\xi}_2)(t + \theta)\mathrm{d}\theta \right\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} + \left(\int_0^{\beta} \left\| \int_{-1}^{0} A_3(\theta)(\xi_1 - \xi_2)(t + \theta)\mathrm{d}\theta \right\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&\le \left(\int_0^{\beta} \left(\int_{-1}^{0} \|A_2(\theta)(\xi_1 - \xi_2)(t + \theta)\|_X \,\mathrm{d}\theta \right)^p \mathrm{d}t \right)^{\frac{1}{p}} + \left(\int_0^{\beta} \left(\int_0^{t} \left\{ \int_{-1}^{0} \|A_3(\theta)(\xi_1 - \xi_2)(\tau + \theta)\|_X \,\mathrm{d}\theta \right\} \mathrm{d}\tau \right)^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&+ \left(\int_0^{\beta} \left(\int_{-1}^{0} \left\|A_2(\theta)(\dot{\xi}_1 - \dot{\xi}_2)(t + \theta)\right\|_X \mathrm{d}\theta \right)^p \mathrm{d}t \right)^{\frac{1}{p}} + \left(\int_0^{\beta} \left(\int_{-1}^{0} \|A_3(\theta)(\xi_1 - \xi_2)(t + \theta)\|_X \,\mathrm{d}\theta \right)^p \mathrm{d}t \right)^{\frac{1}{p}}.
\end{aligned}
$$

Using the fact that $\|A_{2,3}(\theta)x\|_X \le \|A_{2,3}(\theta)\|_{\mathcal{L}(X)}\|x\|_X$, the Hölder's inequality and the assumption (3.2), we obtain

$$
\begin{aligned}
&\|B(\xi_1)(\cdot) - B(\xi_2)(\cdot)\|_{W^{1,p}}([-1,\beta];X) \\
&\qquad \le \left(\int_0^{\beta} \left(\|A_2(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \, \|(\xi_1 - \xi_2)(t + \cdot)\|_{L^p([-1,0];X)} \right)^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&\qquad + \left(\int_0^{\beta} \left(\int_0^{t} \|A_3(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \, \|(\xi_1 - \xi_2)(\tau + \cdot)\|_{L^p([-1,0];X)} \,\mathrm{d}\tau \right)^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&\qquad + \left(\int_0^{\beta} \left(\|A_2(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \, \|(\dot{\xi}_1 - \dot{\xi}_2)(t + \cdot)\|_{L^p([-1,0];X)} \right)^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&\qquad + \left(\int_0^{\beta} \left(\|A_3(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \, \|(\xi_1 - \xi_2)(t + \cdot)\|_{L^p([-1,0];X)} \right)^p \mathrm{d}t \right)^{\frac{1}{p}}.
\end{aligned}
$$

Now, using the fact that for $t \in [0, \beta]$

$$
\begin{aligned}
\|(\xi_1 - \xi_2)(t + \cdot)\|_{L^p([-1,0];X)} &\le \|(\xi_1 - \xi_2)(\cdot)\|_{L^p([-1,\beta];X)}, \\
\|(\dot{\xi}_1 - \dot{\xi}_2)(t + \cdot)\|_{L^p([-1,0];X)} &\le \|(\dot{\xi}_1 - \dot{\xi}_2)(\cdot)\|_{L^p([-1,\beta];X)},
\end{aligned}
$$

we get

$$\begin{aligned}
&\|B(\xi_1)(\cdot) - B(\xi_2)(\cdot)\|_{W^{1,p}([-1,\beta];X)} \\
&\leq \left(\beta\right)^{\frac{1}{p}} \|A_2(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \|(\xi_1 - \xi_2)(\cdot)\|_{L^p([-1,\beta];X)} \\
&+ \left(\frac{\beta^2}{2}\right)^{\frac{1}{p}} \|A_3(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \|(\xi_1 - \xi_2)(\cdot)\|_{L^p([-1,\beta];X)} \\
&+ \left(\beta\right)^{\frac{1}{p}} \|A_2(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \|(\dot{\xi_1} - \dot{\xi_2})(\cdot)\|_{L^p([-1,\beta];X)} \\
&+ \left(\beta\right)^{\frac{1}{p}} \|A_3(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))} \|(\xi_1 - \xi_2)(\cdot)\|_{L^p([-1,\beta];X)} \\
&\leq f(\beta) \|(\xi_1)(\cdot) - (\xi_2)(\cdot)\|_{W^{1,p}([-1,\beta];X)},
\end{aligned}$$

i.e.,

$$\|B(\xi_1)(\cdot) - B(\xi_2)(\cdot)\|_{W^{1,p}([-1,\beta];X)} \leq f(\beta)\|(\xi_1)(\cdot) - (\xi_2)(\cdot)\|_{W^{1,p}([-1,\beta];X)}. \tag{3.6}$$

The positive function $f(\beta)$ tends to 0 as β tends to 0, hence the operation $B : Y(\beta) \to Y(\beta)$ is a contraction mapping for small enough β. This means that there exists a solution of (3.3) on $[0, \beta]$. The existence for all $t \geq 0$ follows from the fact, that the contractivity of B does not depend on the initial data, only on $\|A_2(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))}$ and $\|A_3(\cdot)\|_{L^q([-1,0];\mathcal{L}(X))}$. We can therefore incrementally extend the solution to the whole $\mathbb{R}_0^+$ (with a step equal $\frac{\beta}{2}$ for example). This completes the proof of Lemma 58.

□

From Theorem 14 it follows that in order to prove that the operator $\mathcal{A}$ from (3.3) generates a C_0-semigroup it remains to show that the set $\rho(\mathcal{A})$ is non-empty. This will follow from the considerations below, analogous to the ones presented in [25] for the case where $X = \mathbb{C}^n$.

Proposition 59. [25] *The resolvent of the operator $\mathcal{A}$ is given by*

$$R(\lambda, \mathcal{A}) \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} Ae^{-\lambda} \displaystyle\int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s + (I - Ae^{-\lambda})\Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}} \\ \displaystyle\int_{0}^{\theta} e^{\lambda(\theta - s)}\psi(s)\mathrm{d}s + e^{\lambda\theta}\Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}} \end{pmatrix} \tag{3.7}$$

and $\lambda \in \rho(\mathcal{A})$ if and only if the operator $\Delta_{\mathcal{A}}^{-1}(\lambda) \in \mathcal{L}(X)$ exist, where $D_{\mathcal{A}}$ and $\Delta_{\mathcal{A}}(\lambda)$ are defined as

$$\begin{aligned}
X \ni D_{\mathcal{A}} = {} & x + \lambda e^{-\lambda} A \int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s - \int_{-1}^{0} A_2(s)\psi(s)\mathrm{d}s \\
& - \int_{-1}^{0} \{\lambda A_2(\theta) + A_3(\theta)\} e^{\lambda\theta} \left\{ \int_{0}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s \right\} \mathrm{d}\theta,
\end{aligned}$$

and

$$\mathcal{L}(X) \ni \Delta_{\mathcal{A}}(\lambda) = -\lambda I + \lambda e^{-\lambda}A + \lambda \int_{-1}^{0} e^{\lambda s}A_2(s)\mathrm{d}s + \int_{-1}^{0} e^{\lambda s}A_3(s)\mathrm{d}s.$$

Proof. Consider the equation

$$(\mathcal{A}-\lambda)\begin{pmatrix} y \\ z(\cdot)\end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\mathrm{d}\theta - \lambda z(0) + \lambda A z(-1) \\ \mathrm{d}z(\theta)/\mathrm{d}\theta - \lambda z(\theta) \end{pmatrix} = \begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix} \tag{3.8}$$

From the second line we get

$$z(\theta) = e^{\lambda\theta}z(0) + e^{\lambda\theta}\int_0^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s.$$

This gives

$$\dot{z}(\theta) = \lambda e^{\lambda\theta}z(0) + \lambda e^{\lambda\theta}\int_0^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s + \psi(\theta).$$

Substitute this in the first line of (3.8) and use

$$z(-1) = e^{-\lambda}z(0) - e^{-\lambda}\int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s.$$

By collecting all the terms with $z(0)$ we get

$$\Delta_{\mathcal{A}}(\lambda)z(0) = D_{\mathcal{A}},$$

where $D_{\mathcal{A}}$ is defined as in the statement of the Proposition. Hence

$$z(0) = \Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}},$$

which gives the second line of (3.7). The first line of (3.7) follows from the definition of the domain $D(\mathcal{A})$, i.e., $y = z(0) - Az(-1)$. To see that the existence of the operator $\Delta_{\mathcal{A}}^{-1}(\lambda) \in \mathcal{L}(X)$ is necessary for λ to belong to the set $\rho(\mathcal{A})$ one can consider the vector

$$\begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix} = \begin{pmatrix} x \\ 0\end{pmatrix}.$$

This completes the proof of Proposition 59.

□

Note that the operator $\Delta_{\mathcal{A}}^{-1}(\lambda)$ exists for $\mathrm{Re}(\lambda)$ large enough. This observation combined with Proposition 59 implies a subsequent corollary.

Corollary 60. *The set $\rho(\mathcal{A})$ is non-empty.*

From Theorem 14, Lemma 58, and Corollary 60 we obtain that the operator $\mathcal{A}$ in (3.3) generates a C_0-semigroup.

Theorem 61. *For an arbitrary Banach space X and $p \geq 1$, the operator $\mathcal{A}$ in (3.3) generates a C_0-semigroup in the space $X \times L^p([-1,0];X)$ whenever the condition (3.2) is satisfied.*

The main result of this chapter (Theorem 74) relies on showing that the spectral properties of the operator $\mathcal{A}$, analogously as for the case $X = \mathbb{C}^n$ [25], are similar in some sense to the case when $A(\cdot)_{2,3} \equiv 0$, for which the generator is denoted by $\bar{\mathcal{A}}$. For this particular case Proposition 59 implies the following.

Corollary 62. *For the operator $\bar{\mathcal{A}}$ the resolvent $R(\lambda, \bar{\mathcal{A}})$ takes the form of*

$$R(\lambda, \bar{\mathcal{A}}) \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \dfrac{-x}{\lambda} \\ e^{\lambda\theta} \left(\displaystyle\int_{-1}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s + e^{\lambda}(A - e^{\lambda})^{-1} \left(\int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s + \frac{x}{\lambda} \right) \right) \end{pmatrix} \tag{3.9}$$

Note that (3.9) implies $\lambda \in \sigma(\bar{\mathcal{A}})$ if and only if $e^{\lambda} \in \sigma(A)$ or $\lambda = 0$. This yields

$$\sigma(\bar{\mathcal{A}}) = \bigcup_{k\in\mathbb{Z}} \{\log(\sigma(A)) + 2k\pi i\} \cup \{0\}. \tag{3.10}$$

By $\sigma(A)$ we denote the spectrum of the operator A and by $\log(\sigma(A))$ we mean the principal branch of the logarithm of the set $\sigma(A)$. Further in this chapter we will use a result which holds for the case when the operator A is invertible.

Corollary 63. *For the operator A invertible, the set $\rho(\mathcal{A})$ contains the half-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq \lambda_0\}$ for some $\lambda_0 < 0$.*

Proof. Let us rewrite $\Delta_{\mathcal{A}}(\lambda)$ as

$$\begin{aligned} \Delta_{\mathcal{A}}(\lambda) &= \frac{\lambda}{e^{\lambda}} \left(A - e^{\lambda} + e^{\lambda} \int_{-1}^{0} e^{\lambda s} A_2(s)\mathrm{d}s + \frac{e^{\lambda}}{\lambda} \int_{-1}^{0} e^{\lambda s} A_3(s)\mathrm{d}s \right) \\ &= \frac{\lambda}{e^{\lambda}} \left(A + B(\lambda) \right), 0 \end{aligned}$$

where $\|B(\lambda)\|_{\mathcal{L}(X)} \to 0$ as $\mathrm{Re}(\lambda) \to -\infty$. Recall that, due to Proposition 59, $\lambda \in \rho(\mathcal{A})$ if and only if $\Delta_{\mathcal{A}}^{-1}(\lambda)$ exists. Due to Theorem 45 and the fact that the operator A is invertible, this is the case for large enough $-\mathrm{Re}(\lambda)$. If follows that the half-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \lambda_0\}$ belongs to the set $\rho(\mathcal{A})$ for some $\lambda_0 < 0$.

□

From here on we will analyze a special case of (3.1) with the condition (3.2), namely the case when:

$$p = q = 2, \text{ and the space } X \text{ is a separable Hilbert space (denoted by } H), \text{ and} \tag{A1}$$

$$\log(\sigma(A)) + 2k\pi i \subset \mathrm{Int}O_k \tag{A2}$$

Where $\mathrm{Int}O_k$ are non-overlapping open sets surrounded by the curves $L_k = L_0 + 2k\pi i$ for some fixed regular bounded curve L_0 surrounding the set $\log(\sigma(A))$ such that $\log(\sigma(A)) \cap L_0 = \emptyset$.

Note that (A2) implies the invertibility of the operator A. Such operators are a generalization of invertible matrix operators that appeared in the case of $H = \mathbb{C}^n$ considered in [25]. Note that it follows from Corollary 62 that

$$\sigma(\bar{\mathcal{A}}) \subset \bigcup_{k\in\mathbb{Z}} \mathrm{Int}O_k \cup \{0\}.$$

From here on we will denote the space $H \times L^2([-1,0];H)$, similarly as for the case of $H = \mathbb{C}^n$, as

$$H \times L^2([-1,0];H) \equiv M_2.$$

The space M_2 is equipped with a scalar product given by

$$\left\langle \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix}, \begin{pmatrix} y \\ \phi(\cdot) \end{pmatrix} \right\rangle_{M_2} = \langle x, y\rangle_H + \int_{-1}^{0} \langle \psi(\theta), \phi(\theta)\rangle_H \mathrm{d}\theta, \tag{3.11}$$

which induces a norm on M_2 of the form

$$\left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2} = \left(\|x\|_H^2 + \int_{-1}^{0} \|\psi(\theta)\|_H^2 \mathrm{d}\theta \right)^{\frac{1}{2}}. \tag{3.12}$$

The assumption (A1) yields the space M_2 a Hilbert space, analogously as for the finite-dimensional case [25]. This is due to the fact that it is a product space of two Hilbert spaces (see Definition 33). We will also put some additional constraints on the operator-valued functions $A_{2,3}(\theta) : H \to H$, $\theta \in [-1,0]$, namely that for each $\theta \in [-1,0]$ the operator-valued functions in question are Hilbert-Schmidt operators and their Hilbert-Schmidt norm is square integrable over $\theta \in [-1,0]$ (see Theorem 40), i.e.,

$$\int_{-1}^{0} \mathrm{Tr}\left(A_{2,3}^*(\theta)A_{2,3}(\theta)\right) \mathrm{d}\theta < \infty. \tag{3.13}$$

Recall now Theorem 40 and Definition 33. It follows that the space consisting of all strongly measurable operator-valued functions $K(\cdot) \in L^2([-1,0];\mathcal{L}_{HS}(H))$ satisfying

$$\int_{-1}^{0} \mathrm{Tr}\left(K^*(\theta)K(\theta)\right) \mathrm{d}\theta < \infty, \tag{3.14}$$

is a Hilbert space with the scalar product given by

$$\langle K(\cdot), M(\cdot)\rangle \equiv \int_{-1}^{0} \mathrm{Tr}\left(M^*(\theta)K(\theta)\right) \mathrm{d}\theta. \tag{3.15}$$

The space $\mathcal{L}_{HS}(H)$ is separable whenever H is separable (see Section 1.4 of Chapter 1), and so is $L^2\left([-1,0];\mathcal{L}_{HS}(H)\right)$, as follows from the considerations in Lemma 64, given below. Also note that it follows from Proposition 42 that

$$\int_{-1}^{0} \|K(\theta)\|_{\mathcal{L}(H)}^2 \mathrm{d}\theta \le \int_{-1}^{0} \|K(\theta)\|_{\mathcal{L}_{HS}(H)}^2 \mathrm{d}\theta, \tag{3.16}$$

hence

$$L^2([-1,0];\mathcal{L}_{HS}(H)) \subset L^2([-1,0];\mathcal{L}(H)). \tag{3.17}$$

The property (3.17) allows to consider the operator-valued functions $A_{2,3}(\cdot) \in L^2([-1,0];\mathcal{L}_{HS}(H))$ such that the condition (A1) is satisfied, allowing to use ideas and techniques from general Hilbert space theory and Hilbert space operator theory to both the space M_2 and the operator-valued functions $A_{2,3}(\cdot)$ that define (3.1). The Hilbert-Schmidt operators are a natural extension of matrix operators used for the case when $H = \mathbb{C}^n$ in [22–25] (see section 1.4 in Chapter 1).

Having stated the assumptions taken for the studied system, we proceed with the obtained results. We begin with a rather intuitive property of generalized $L^2[-1,0]$ Hilbert spaces. The following technical lemma shows that there holds a Fourier decomposition property for any function $\psi(\cdot) \in L^2([-1,0];H)$.

Lemma 64. *Let $\psi(\cdot) \in L^2([-1,0];H)$, where H is a separable Hilbert space. Then it holds that*

$$\psi(\cdot) = \sum_{k\in\mathbb{Z}} e^{2k\pi i(\cdot)} \int_{-1}^{0} e^{-2k\pi is}\psi(s)\mathrm{d}s, \tag{3.18}$$

and

$$\|\psi(\cdot)\|^2_{L^2([-1,0];H)} = \sum_{k\in\mathbb{Z}} \left\| \int_{-1}^{0} e^{-2k\pi is}\psi(s)\mathrm{d}s \right\|^2_H . \tag{3.19}$$

Proof. Note first that the space $L^2([-1,0];H)$ is a Hilbert space with the scalar product given by

$$\langle\psi(\cdot),\phi(\cdot)\rangle_{L^2([-1,0];H)} = \int_{-1}^{0} \langle\psi(\theta),\phi(\theta)\rangle_H \mathrm{d}\theta.$$

Since the space H is separable, any vector function $\psi(\cdot) \in L^2([-1,0];H)$ admits the following orthonormal decomposition

$$\psi(\cdot) = \sum_{k\in\mathbb{Z},n\in\mathbb{N}} \alpha_{k,n} e^{2k\pi i(\cdot)} h_n, \qquad \sum_{k\in\mathbb{Z},n\in\mathbb{N}} |\alpha_{k,n}|^2 = \|\psi(\cdot)\|^2_{L^2([-1,0];H)}, \tag{3.20}$$

where $\{h_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H. This follows from the facts, that each simple function can be approximated by a finite sum of the form as in (3.20) and that simple functions are dense in $L^2([-1,0];H)$ since $L^2([-1,0];H)$ is a Bochner space. It follows the coefficients $\alpha_{k,n}$ are of the form

$$\begin{aligned} \alpha_{k,n} &= \langle\psi(\cdot), e^{2k\pi i(\cdot)}h_n\rangle_{L^2([-1,0];H)} \\ &= \int_{-1}^{0} \langle\psi(s), e^{2k\pi is}h_n\rangle_H \mathrm{d}s \\ &= \int_{-1}^{0} e^{-2k\pi is}\langle\psi(s), h_n\rangle_H \mathrm{d}s \end{aligned} \tag{3.21}$$

It follows from (3.20) and (3.21), that

$$\psi(\cdot) = \sum_{k\in\mathbb{Z}} e^{2k\pi i(\cdot)} \sum_{n\in\mathbb{N}} \left\{ \int_{-1}^{0} e^{-2k\pi is}\langle\psi(s), h_n\rangle_H \mathrm{d}s \right\} h_n. \tag{3.22}$$

Now, since $\{h_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H, it follows that, for fixed $s \in [-1, 0]$, the series

$$\sum_{n\in\mathbb{N}} \langle \psi(s), h_n \rangle_H h_n$$

converges to $\psi(s)$ pointwise, i.e.,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \langle \psi(s), h_n \rangle_H h_n = \psi(s), \quad \text{for all } s \in [-1, 0]. \tag{3.23}$$

Due to the same arguments and the use of the Bessel's inequality, we get

$$\left\| \sum_{n=1}^{N} \langle \psi(s), h_n \rangle_H h_n \right\|_H \leq \|\psi(s)\|_H\,, \quad \text{for all } N \in \mathbb{N} \text{ and } s \in [-1, 0]. \tag{3.24}$$

Now, let us rewrite (3.22) as

$$\psi(\cdot) = \sum_{k\in\mathbb{Z}} e^{2k\pi i(\cdot)} \lim_{N\to\infty} \sum_{n=1}^{N} \left\{ \int_{-1}^{0} e^{-2k\pi is} \langle \psi(s), h_n \rangle_H h_n \mathrm{d}s \right\}.$$

By taking into consideration (3.23) and (3.24) and using the Dominated Convergence Theorem for Bochner integrals (Theorem 32), we can move the summation sign under the integral to obtain

$$\psi(\cdot) = \sum_{k\in\mathbb{Z}} e^{2k\pi i(\cdot)} \int_{-1}^{0} e^{-2k\pi is} \psi(s) \mathrm{d}s,$$

thus proving (3.18). The equality (3.19) follows from the fact that the set of functions $\{e^{2k\pi i(\cdot)}\}_{k\in\mathbb{Z}}$ forms an orthonormal set in the $L^2([-1, 0], \mathbb{C})$ space. This completes the proof of Lemma 64.

□

Lemma 64 will be of use several times when proving the existence of a sequence of $\mathcal{A}$-invariant subspaces which constitute a Riesz basis in the space M_2 for both the perturbed and unperturbed cases. The existence of a Riesz basis consisting of Riesz projections for the unperturbed case ($A(\cdot)_{2,3} \equiv 0$), is shown in Lemma 69. Before we proceed however, we need to prove a few more lemmas of a technical nature. Note that by K^* we denote the adjoint of the operator K and that in this work by the term *projection operator* we mean a projection operator which is bounded.

Lemma 65. *Let $\{R_k\}_{k\in\mathbb{Z}}$ be a family of orthogonal projection operators (for which it holds that $R_k = R_k^*, \quad k \in \mathbb{Z}$) on a Hilbert space H. Let $\{S_k\}_{k\in\mathbb{Z}}$ be a family of projection operators such that*

$$\sum_{k\in\mathbb{Z}} \|R_k - S_k\|^2_{\mathcal{L}(H)} < \infty.$$

Then the two families of subspaces $\{R_k H\}_{k\in\mathbb{Z}}$ and $\{S_k H\}_{k\in\mathbb{Z}}$ are quadratically close (see Definition 23), i.e.,

$$\sum_{k\in\mathbb{Z}} \|R_k^{Ort} - S_k^{Ort}\|^2_{\mathcal{L}(H)} = \sum_{k\in\mathbb{Z}} \|R_k - S_k^{Ort}\|^2_{\mathcal{L}(H)} < \infty,$$

where we denote the orthogonal projection onto the space $H_k = S_k H$ by S_k^{Ort} with an analogous notation for the orthogonal projection onto the space $R_k H$.

Proof. Note that by assumption it holds that $R_k = R_k^{Ort}$. We will show that

$$\|R_k^{Ort} - S_k^{Ort}\|_{\mathcal{L}(H)} = \|R_k - S_k^{Ort}\|_{\mathcal{L}(H)} \leq c_k, \tag{3.25}$$

where $\sum_{k\in\mathbb{Z}} c_k^2 < \infty$. Keeping in mind that $R_k = R_k^*$ (since each R_k is an orthogonal projection), it holds by assumption that

$$\|R_k - S_k\|_{\mathcal{L}(H)} = \|R_k^* - S_k^*\|_{\mathcal{L}(H)} = \|R_k - S_k^*\|_{\mathcal{L}(H)} \leq d_k, \tag{3.26}$$

where $\sum_{k\in\mathbb{Z}} d_k^2 < \infty$. From (3.26) we get

$$\|S_k - S_k^*\|_{\mathcal{L}(H)} \leq \|S_k - R_k\|_{\mathcal{L}(H)} + \|R_k - S_k^*\|_{\mathcal{L}(H)} \leq 2d_k \tag{3.27}$$

where $\sum_{k\in\mathbb{Z}} d_k^2 < \infty$. Let us decompose S_k as

$$S_k = S_k S_k^{Ort} + S_k(I - S_k^{Ort}), \tag{3.28}$$

and note that $S_k S_k^{Ort} = S_k^{Ort}$, i.e., $S_k S_k^{Ort}$ is self-adjoint. Keeping in mind that $(I - S_k^{Ort})$ is also self-adjoint, we get

$$S_k^* = (S_k S_k^{Ort})^* + (S_k(I - S_k^{Ort}))^* = (S_k S_k^{Ort}) + (I - S_k^{Ort})S_k^*.$$

This yields

$$S_k - S_k^* = S_k(I - S_k^{Ort}) - (I - S_k^{Ort})S_k^*.$$

Now since, it holds for any $x \in H$,

$$S_k(I - S_k^{Ort})x \in H_k,$$

and

$$(I - S_k^{Ort})S_k^* x \in H_k^{\perp},$$

due to (3.27), we obtain

$$\|S_k(I - S_k^{Ort})x\|_H^2 \leq \|S_k(I - S_k^{Ort})x\|_H^2 + \|(I - S_k^{Ort})S_k^* x\|_H^2 = \|(S_k - S_k^*)x\|_H^2 \leq 4d_k^2\|x\|_H^2,$$

where $\sum_{k\in\mathbb{Z}} d_k^2 < \infty$. This implies

$$\|S_k(I - S_k^{Ort})\| \leq 2d_k, \tag{3.29}$$

Now, using (3.28) and (3.29), we obtain

$$\|S_k - S_k^{Ort}\|_{\mathcal{L}(H)} = \|S_k - S_k S_k^{Ort}\|_{\mathcal{L}(H)} = \|S_k(I - S_k^{Ort})\|_{\mathcal{L}(H)} \leq 2d_k,$$

where $\sum_{k\in\mathbb{Z}} d_k^2 < \infty$. Combining the above inequality with (3.26) yields

$$\|R_k - S_k^{Ort}\|_{\mathcal{L}(H)} \le \|R_k - S_k\|_{\mathcal{L}(H)} + \|S_k - S_k^{Ort}\|_{\mathcal{L}(H)} \le d_k + 2d_k = c_k,$$

where $\sum_{k\in\mathbb{Z}} c_k^2 < \infty$. This proves (3.25), thus completing the proof of Lemma 65.

□

Lemma 66. *Let $\{R_k\}_{k\in\mathbb{Z}}$ be a family of mutually orthogonal ($R_jR_k = \delta_{j,k}R_k$) projection operators on a Hilbert space H. Then for each k the minimal angle (see Definition 25) between the space R_kH and the closed linear hull $\overline{\operatorname{span}\{R_jH,\, j\neq k\}}$ is positive.*

Proof. Let k be fixed, $N \in \mathbb{N}$ be arbitrary and let

$$x = \sum_{|n|\le N,\, n\neq k} \alpha_n x_n, \quad x_n = R_n y_n, \quad y_n \in H.$$

Then, since $R_jR_k = \delta_{j,k}R_k$, we get that $x \in \ker R_k$. Thus, due to the density of the elements of the form x in $\overline{\operatorname{span}\{R_jH,\, j\neq k\}}$ and the boundedness of the operator R_k, the closed linear hull in question is a subspace of $\ker R_k$. It follows from the definition of the minimal angle $0 \le \phi \le \frac{\pi}{2}$ between subspaces $\mathfrak{U}, \mathfrak{V}$,

$$\cos\phi(\mathfrak{U},\mathfrak{V}) = \sup_{x\in\mathfrak{U}, y\in\mathfrak{V},\, \|x\|=\|y\|=1} |\langle x, y\rangle|,$$

that in order to prove the thesis, it suffices to show that the minimal angle between the subspaces R_kH and $\ker R_k$ is positive. Assume the contrary, i.e., there exists a sequence of pairs $(x_n, y_n) \in R_kH \times \ker R_k$, $\|x_n\| = \|y_n\| = 1$, such that

$$\lim_{n\to\infty} |\langle x_n, y_n\rangle| = 1. \tag{3.30}$$

Note that by multiplying the scalar product $\langle x_n, y_n\rangle$ by $\frac{\overline{\langle x_n, y_n\rangle}}{|\langle x_n, y_n\rangle|}$ we can see that (3.30) implies the existence of a sequence of pairs (we do not change the notation to maintain clarity) $(x_n, y_n) \in R_kH \times \ker R_k$, $\|x_n\| = \|y_n\| = 1$ such that

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = 1.$$

It is easy to see that in such a case we obtain

$$\lim_{n\to\infty} \langle x_n - y_n, x_n - y_n\rangle = \lim_{n\to\infty} \|x_n\|^2 + \lim_{n\to\infty} \|y_n\|^2 - \lim_{n\to\infty} 2\mathrm{Re}\langle x_n, y_n\rangle = 0.$$

Thus $\lim_{n\to\infty} \|x_n - y_n\| = 0$ whenever (3.30) holds. This implies, due to the boundedness of the projection operator R_k, that

$$1 = \|x_n\| = \|R_kx_n\| = \|R_k(x_n - y_n)\| \to 0, \text{ as } n\to\infty.$$

This contradiction shows that (3.30) cannot hold, which ends the proof of Lemma 66.

$\square$

Remark 67. *Note that any family of Riesz projections corresponding to disjoint subsets of the spectrum (see Definition 16) satisfies the assumptions of Lemma 66. Also note that a family of mutually orthogonal projection operators* $\{R_k\}_{k\in\mathbb{Z}}$ *will remain a family of mutually orthogonal projection operators if we change the norm to an equivalent one.*

The following Lemma shows that the *non-zero* property of the minimal angle between subspaces of a Hilbert space is invariant w.r.t. the change to a scalar product is such a way that the new norm induced by the new scalar product is equivalent to the original norm. This property will be used in the proof of Theorem 74 at the end of this chapter.

Lemma 68. *Let a new scalar product on a given Hilbert space* H *induce a norm on the space* H *which is equivalent to the norm induced by the original scalar product. If the minimal angle between two subspaces* $\mathfrak{U}, \mathfrak{V}$ *is positive w.r.t. the original scalar product, than it remains so in w.r.t. the new scalar product.*

Proof. Assume the contrary, i.e, the minimal angle with respect to the original scalar product $\langle\cdot,\cdot\rangle_1$ is positive, while it equals 0 w.r.t. the new scalar product $\langle\cdot,\cdot\rangle_2$. From the considerations in Lemma 66 it follows the existence of a normed, w.r.t. the new norm $\|\cdot\|_2$, sequence of pairs of elements $(x_n, y_n) \in \mathfrak{U}\times\mathfrak{V}$ such that

$$\|x_n - y_n\|_2 \to 0, \quad \text{as } n \to \infty.$$

Consider the normed, w.r.t. the original norm $\|\cdot\|_1$, sequence of pairs of elements of the form

$$\frac{x_n}{\|x_n\|_1}, \frac{y_n}{\|y_n\|_1}.$$

Note that, since the norms are equivalent, it holds that

$$\frac{1}{C^2} \le \frac{1}{\|x_n\|_1\|y_n\|_1} \le \frac{1}{c^2}, \quad n \ge 0,$$

$$\|x_n\|_1 \le C, \quad n \ge 0$$

for some $c, C > 0$. We thus obtain

$$\begin{aligned}
&\left\|\frac{x_n}{\|x_n\|_1} - \frac{y_n}{\|y_n\|_1}\right\|_1 = \left\|\frac{x_n\|y_n\|_1 - x_n\|x_n\|_1 + x_n\|x_n\|_1 - y_n\|x_n\|_1}{\|x_n\|_1\|y_n\|_1}\right\|_1 \\
&\le \frac{1}{c^2}\big\|\{x_n\|y_n\|_1 - x_n\|x_n\|_1 + x_n\|x_n\|_1 - y_n\|x_n\|_1\}\big\|_1 \le \frac{C}{c^2}|\|y_n\|_1 - \|x_n\|_1| + \frac{C}{c^2}\|x_n - y_n\|_1 \\
&\le \frac{2C}{c^2}\|x_n - y_n\|_1 \le D\|x_n - y_n\|_2 \to 0 \text{ as } n \to \infty
\end{aligned}$$

for some $D > 0$. This implies the existence of a sequence of pairs $(\hat{x}_n, \hat{y}_n) \in \mathfrak{U}\times\mathfrak{V}$ of normed w.r.t. the norm $\|\cdot\|_1$ elements such that

$$\|\hat{x}_n - \hat{y}_n\|_1 \to 0, \quad \text{as } n \to \infty.$$

It follows that

$$\langle \hat{x}_n - \hat{y}_n, \hat{x}_n - \hat{y}_n \rangle_1 = \|\hat{x}_n\|_1^2 + \|\hat{y}_n\|_1^2 - 2\mathrm{Re}\langle \hat{x}_n, \hat{y}_n \rangle_1 \to 0,$$

and subsequently,

$$|\langle \hat{x}_n, \hat{y}_n \rangle_1| \to 1, \quad \text{as } n \to \infty.$$

Which means that the minimal angle between the subspaces $\mathfrak{U}$ and $\mathfrak{Y}$ equals 0 w.r.t the original scalar product $\langle \cdot, \cdot \rangle_1$, which contradicts the assumption, thus proving Lemma 68.

Now we will prove the existence of a Riesz basis consisting of Riesz projections for the case of $A(\cdot)_{2,3} \equiv 0$ for which the corresponding operator in (3.3) will be denoted by $\bar{\mathcal{A}}$.

Lemma 69. *Consider the equation (3.3) with $A(\cdot)_{2,3} \equiv 0$ and let $L_k = L_0 + 2k\pi i$ be a family of regular bounded curves surrounding the sets $\{\log(\sigma(A)) + 2k\pi i\}_{k\in\mathbb{Z}}$ (see Corollary 62) such that $L_0 \cap \log(\sigma(A)) = \emptyset$ and that the bounded subsets O_k of $\mathbb{C}$ enclosed by each L_k are such that $\mathrm{Int}O_k \cap \mathrm{Int}O_l = \emptyset$ for $k \neq l$. Then the subspaces of M_2 which are the images of the Riesz projections $\bar{P}_k$, $k \in \mathbb{Z}$, of the operator $\bar{\mathcal{A}}$ associated with the curves L_k constitute a Riesz basis of $\bar{\mathcal{A}}$-invariant subspaces of the space M_2.*

Proof. The existence of such a family of curves L_k follows from the assumption (A2) on the operator A. Note that each L_k, due to Corollary 62, is a subset of $\rho(\bar{\mathcal{A}})$ and that $\sigma(\bar{\mathcal{A}}) \subset \bigcup_{k\in\mathbb{Z}} \mathrm{Int}O_k \cup \{0\}$. Without the loss of generality we can assume that $0 \in O_0$. Consider the family of Riesz projections $\bar{P}_k$ corresponding to the operator $\bar{\mathcal{A}}$ and the curves L_k. It follows from (3.9) that

$$\bar{P}_k \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = -\frac{1}{2\pi i} \oint_{L_0+2k\pi i} \begin{pmatrix} \dfrac{-x}{\lambda} \\ e^{\lambda\theta}\left(\int_{-1}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s + e^{\lambda}(A - e^{\lambda})^{-1}\left(\int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s + \dfrac{x}{\lambda}\right)\right) \end{pmatrix} \mathrm{d}\lambda.$$

Due to the Cauchy's integral formula, the above equation reduces to

$$\bar{P}_k \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \delta_{0,k} x \\ -\dfrac{1}{2\pi i} \oint_{L_0+2k\pi i} e^{\lambda\theta} e^{\lambda}(A - e^{\lambda})^{-1}\left(\int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s + \dfrac{x}{\lambda}\right)\mathrm{d}\lambda \end{pmatrix}, \tag{3.31}$$

where $\delta_{0,k}$ denotes the Kronecker delta. After a change of variables of the form $\mu = e^{\lambda}$, we obtain

$$\bar{P}_k \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \delta_{0,k} x \\ -\dfrac{1}{2\pi i} e^{2k\pi i\theta} \oint_{e^{L_0}} \mu^{\theta}(A - \mu)^{-1}\left(\int_{-1}^{0} e^{-2k\pi i s}\mu^{-s}\psi(s)\mathrm{d}s + \dfrac{x}{\log(\mu) + 2k\pi i}\right)\mathrm{d}\mu \end{pmatrix}, \tag{3.32}$$

where the curve e^{L_0} surrounds the set $\sigma(A)$. Using the Dunford calculus for bounded operators (see

Definition 46 and Theorem 47), from (3.32) we get

$$\bar{P}_k \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \delta_{0,k}x \\ e^{2k\pi i\theta} A^\theta \int_{-1}^{0} e^{-2k\pi is} A^{-s}\psi(s)\mathrm{d}s - \frac{1}{2\pi i} e^{2k\pi i\theta} \oint\limits_{e^{L_0}} \mu^\theta (A-\mu)^{-1} \frac{x}{\log(\mu)+2k\pi i}\mathrm{d}\mu \end{pmatrix}. \tag{3.33}$$

We will now show that the family of subspaces generated by the projection operators of the form

$$\bar{Q}_k \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \equiv \begin{pmatrix} \delta_{0,k}x \\ e^{2k\pi i\theta} A^\theta \int_{-1}^{0} e^{-2k\pi is} A^{-s}\psi(s)\mathrm{d}s \end{pmatrix} \tag{3.34}$$

constitutes a Riesz basis of subspaces of the space M_2. Due to Lemma 64 for arbitrary $\psi(\cdot)$ and x one obtains

$$\sum_{k\in\mathbb{Z}} \bar{Q}_k \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} x \\ A^{(\cdot)}A^{-(\cdot)}\psi(\cdot) \end{pmatrix} = \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix}.$$

Also, since the sum

$$\sum_{k\in\mathbb{Z}} e^{2k\pi i\theta} \int_{-1}^{0} e^{-2k\pi is} A^{-s}\psi(s)\mathrm{d}s$$

is the Fourier expansion of the vector function $A^{-(\cdot)}\psi(\cdot)$, it remains so for any permutation k' of the indices k. It follows that

$$\sum_{k'\in\mathbb{Z}} \bar{Q}_{k'} \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \tag{3.35}$$

for any permutation k' of the indices k and for arbitrary $\psi(\cdot)$ and x. The application of Theorem 20 yields that the set of subspaces generated by the projection operators $\{\bar{Q}_k\}_{k\in\mathbb{Z}}$ constitutes a Riesz basis of subspaces of the space M_2. In order for the set of subspaces generated by the projections $\{\bar{P}_k\}_{k\in\mathbb{Z}}$ to be a Riesz basis, the subspaces generated by the projections $\{\bar{P}_k\}_{k\in\mathbb{Z}}$ need to form a complete system. This can be seen from the following argument. Let $\begin{pmatrix} x_0 \\ \psi_0(\cdot) \end{pmatrix} \in M_2$, then, due to (3.33), we obtain

$$\begin{pmatrix} x_0 \\ \psi_0(\cdot) \end{pmatrix} - \bar{P}_0 \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_1(\cdot) \end{pmatrix}$$

for some $\psi_1(\cdot) \in L^2([-1,0];H)$. Now, due to (3.33) and (3.35), we obtain

$$\sum_{k\in\mathbb{Z}} \bar{P}_k \begin{pmatrix} 0 \\ \psi_1(\cdot) \end{pmatrix} = \sum_{k\in\mathbb{Z}} \bar{Q}_k \begin{pmatrix} 0 \\ \psi_1(\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_1(\cdot) \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} x_0 \\ \psi_0(\cdot) \end{pmatrix} = \bar{P}_0 \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \sum_{k\in\mathbb{Z}} \bar{P}_k \begin{pmatrix} 0 \\ \psi_1(\cdot) \end{pmatrix}. \tag{3.36}$$

Thus the subspaces $\{\bar{P}_k M_2\}_{k\in\mathbb{Z}}$ form a complete system.

Now, from (3.33) and (3.34) we get

$$\left\|(\bar{P}_k-\bar{Q}_k)\begin{pmatrix}x\\ \psi(\cdot)\end{pmatrix}\right\|_{M_2} = \left\|\begin{pmatrix}0\\ \frac{1}{2\pi i}e^{2k\pi i\theta}\oint\limits_{e^{L_0}}\mu^{\theta}(A-\mu)^{-1}\frac{x}{\log(\mu)+2k\pi i})\mathrm{d}\mu\end{pmatrix}\right\|_{M_2} \le \left(\frac{C_2}{|k|+1}\right)\|x\|_H \le \left(\frac{C_2}{|k|+1}\right)\left\|\begin{pmatrix}x\\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \tag{3.37}$$

where we have used the fact that $\dfrac{1}{|\log(\mu)+2k\pi i|}\le\dfrac{C_1}{|k|+1}$ on e^{L_0} for some $C_1>0$ independent of k and the uniform boundedness of $\left\|\mu^{\theta}(A-\mu)^{-1}\right\|$ on $e^{L_0}\times[-1,0]$. Now, (3.37) implies that

$$\sum_{k\in\mathbb{Z}}\left\|\bar{P}_k-\bar{Q}_k\right\|^2_{\mathcal{L}(M_2)}\le C_2^2\sum_{k\in\mathbb{Z}}\left(\frac{1}{|k|+1}\right)^2<\infty. \tag{3.38}$$

Let us introduce a new scalar product $\langle\cdot,\cdot\rangle_{M_{2,1}}$ on M_2 which equips the space with a norm equivalent to the initial one (cf. (3.11)) by

$$\left\langle\begin{pmatrix}x\\ \psi(\cdot)\end{pmatrix},\begin{pmatrix}y\\ \phi(\cdot)\end{pmatrix}\right\rangle_{M_{2,1}}\equiv\left\langle\begin{pmatrix}x\\ A^{-(\cdot)}\psi(\cdot)\end{pmatrix},\begin{pmatrix}y\\ A^{-(\cdot)}\phi(\cdot)\end{pmatrix}\right\rangle_{M_2}$$

The norm equivalence comes from the fact that the bounded operator $A^{-(\cdot)}$ on $L^2([-1,0];H)$ is invertible with a bounded inverse $A^{(\cdot)}$. This can be seen by writing (cf. (3.12))

$$\left\|\begin{pmatrix}x\\ A^{-(\cdot)}\psi(\cdot)\end{pmatrix}\right\|^2_{M_{2,1}}\equiv\|x\|^2_H+\|A^{-(\cdot)}\psi(\cdot)\|^2_{L^2([-1,0];H)}.$$

It follows from the form of subspaces generated by the projections $\{\bar{Q}_k\}_{k\in\mathbb{Z}}$, which form a complete set of subspaces (see (3.34)), that with respect to the new scalar product $\langle\cdot,\cdot\rangle_{M_{2,1}}$ these subspaces are mutually orthogonal and the projections $\{\bar{Q}_k\}_{k\in\mathbb{Z}}$ are orthogonal projections. Since the norm given by the new scalar product is equivalent to the original norm, it follows from (3.38) and Lemma 65, that the subspaces generated by the projection operators $\{\bar{P}_k\}_{k\in\mathbb{Z}}$ are quadratically close to a complete set of orthogonal subspaces generated by $\{\bar{Q}_k\}_{k\in\mathbb{Z}}$ w.r.t. the new norm. The fact that the projection operators $\{\bar{P}_k\}_{k\in\mathbb{Z}}$ generate a Riesz basis of subspaces of the space M_2 equipped with the norm $\|\cdot\|_{M_{2,1}}$, and thus w.r.t. the original norm (since the norms are equivalent, see Theorem 20), follows from the observation that the set of subspaces $\{\bar{P}_kM_2\}_{k\in\mathbb{Z}}$ is complete, Theorem 24 followed by Remark 26, and the fact that the minimal angle between the subspace $\bar{P}_kM_2$ and the closed linear hull of the rest of subspaces $\bar{P}_kM_2$, $(j\neq k)$ is positive for each k. The last property is due to Lemma 66 followed by the Remark 67, and the fact that

$$\bar{P}_j\bar{P}_k=\delta_{j,k}\bar{P}_k.$$

which holds since the operators $\bar{P}_k$ are Riesz projections corresponding to disjoint subsets of the spectrum of the operator $\bar{\mathcal{A}}$. The fact that the subspaces $\{\bar{P}_kM_2\}_{k\in\mathbb{Z}}$ are $\bar{\mathcal{A}}$-invariant follows from the fact that they are the images of Riesz projections of the operator $\bar{\mathcal{A}}$. This observation completes the proof of Lemma 69.

□

Having proved the existence of a Riesz basis consisting of Riesz projections for the unperturbed case ($A(\cdot)_{2,3} \equiv 0$) we will proceed to some technical lemmas necessary to prove the existence of a Riesz basis constructed from Riesz projections for the case of a non-zero perturbation. The lemma presented below can be thought of as a development of Lemma 64. In the notation below we omit the operator I and write shortly $f(s)$ instead of $If(s)$.

Lemma 70. *Let $K(\cdot) \in L^2([-1,0];\mathcal{L}_{HS}(H))$ and J_0 be a compact set in $\mathbb{C}$. Then it holds that*

$$\sup_{\lambda\in J_k}\left\|\int_{-1}^{0} e^{\lambda s}K(s)\mathrm{d}s\right\|_{\mathcal{L}(H)} \le \zeta_k, \text{ where } \sum_{k\in\mathbb{Z}}{\zeta_k}^2 < \infty \text{ and } J_k = J_0 + 2k\pi i.$$

Proof. Let us rewrite $\int_{-1}^{0} e^{\lambda s}K(s)\mathrm{d}s$ as

$$\int_{-1}^{0} e^{\lambda s}K(s)\mathrm{d}s = \int_{-1}^{0} e^{\hat{\lambda} s}e^{2k\pi i s}K(s)\mathrm{d}s,$$

where $\hat{\lambda} \in J_0$. By the means of integration by parts for Bochner integrals (see Theorem 36) we obtain

$$\begin{aligned}
\left\|\int_{-1}^{0} e^{\hat{\lambda} s}e^{2k\pi i s}K(s)\mathrm{d}s\right\|_{\mathcal{L}(H)} &= \left\|\int_{-1}^{0}\left(\int_{-1}^{s}\hat{\lambda}e^{\hat{\lambda}t}\mathrm{d}t + e^{-\hat{\lambda}}\right)e^{2k\pi i s}K(s)\mathrm{d}s\right\|_{\mathcal{L}(H)} \\
&= \left\|\int_{-1}^{0}\hat{\lambda}e^{\hat{\lambda}t} + e^{-\hat{\lambda}}\mathrm{d}t\int_{-1}^{0}e^{2k\pi i t}K(t)\mathrm{d}t - \int_{-1}^{0}\hat{\lambda}e^{\hat{\lambda}s}\int_{-1}^{s}e^{2k\pi i t}K(t)\mathrm{d}t\mathrm{d}s\right\|_{\mathcal{L}(H)} \\
&\le C\left(\left\|\int_{-1}^{0}e^{2k\pi i s}K(s)\mathrm{d}s\right\|_{\mathcal{L}(H)} + \int_{-1}^{0}\left\|\int_{-1}^{s}e^{2k\pi i t}K(t)\mathrm{d}t\right\|_{\mathcal{L}(H)}\mathrm{d}s.\right)
\end{aligned} \tag{3.39}$$

The last step is due to boundedness of $\hat{\lambda}e^{\hat{\lambda}s}$ and $e^{-\hat{\lambda}}$ on the set set $J_0 \times [-1,0]$, and the fact that the norm of a Bochner integral is less or equal than the integral of the norm of the integrand (see Proposition 31). Note that C does not depend on $\hat{\lambda}$. Recall now (see Proposition 42) that the Hilbert-Schmidt norm dominates the operator norm, i.e.,

$$\|K\|_{\mathcal{L}(H)} \le \|K\|_{\mathcal{L}_{HS}(H)}. \tag{3.40}$$

Note that for each $s \in [-1,0]$, due to Lemma 64 it holds that

$$\int_{-1}^{s} e^{2k\pi i t}K(t)\mathrm{d}t = \int_{-1}^{0} e^{2k\pi i t}K(t)\,\chi_{[0,s]}(t)\mathrm{d}t \in \mathcal{L}_{HS}(H),$$

where $\chi_{[0,s]}(\cdot)$ denotes the characteristic function of the interval $[0,s]$. Hence, for all $s \in [-1,0]$

$$\left\| \int_{-1}^{s} e^{2k\pi it} K(t)\mathrm{d}t \right\|_{\mathcal{L}_{HS}(H)}$$

is well-defined. From (3.39) and (3.40) we get

$$\left\| \int_{-1}^{0} e^{\hat{\lambda}s} e^{2k\pi is} K(s)\mathrm{d}s \right\|_{\mathcal{L}(H)} \leq C\left(\left\| \int_{-1}^{0} e^{2k\pi is} K(s)\mathrm{d}s \right\|_{\mathcal{L}_{HS}(H)} + \int_{-1}^{0} \left\| \int_{-1}^{s} e^{2k\pi it} K(t)\mathrm{d}t \right\|_{\mathcal{L}_{HS}(H)} \mathrm{d}s \right). \tag{3.41}$$

Now, since $K(\cdot)$ belongs to the Hilbert space $L^2([-1,0];\mathcal{L}_{HS}(H))$, the first term on the r.h.s. of the inequality belongs to l^2 due to Lemma 64. It remains to estimate the sum of squares of the second term on the r.h.s. of (3.41). First we observe that, due to the Hölder's inequality, we get

$$\sum_{k\in\mathbb{Z}} \left(\int_{-1}^{0} \left\| \int_{-1}^{s} e^{2k\pi it} K(t)\mathrm{d}t \right\|_{\mathcal{L}_{HS}(H)} \mathrm{d}s \right)^2 \leq \sum_{k\in\mathbb{Z}} \int_{-1}^{0} \left\| \int_{-1}^{s} e^{2k\pi it} K(t)\mathrm{d}t \right\|_{\mathcal{L}_{HS}(H)}^2 \mathrm{d}s. \tag{3.42}$$

Notice that the partial sums of the infinite sum on the r.h.s. of (3.42) are non-decreasing for all $s \in [-1,0]$. Due to the Monotone Convergence Theorem we can move the summation under the integral, thus getting

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left(\int_{-1}^{0} \left\| \int_{-1}^{s} e^{2k\pi it} K(t)\mathrm{d}t \right\|_{\mathcal{L}_{HS}(H)} \mathrm{d}s \right)^2 &\leq \int_{-1}^{0} \sum_{k\in\mathbb{Z}} \left\| \int_{-1}^{s} e^{2k\pi it} K(t)\mathrm{d}t \right\|_{\mathcal{L}_{HS}(H)}^2 \mathrm{d}s \\ &= \int_{-1}^{0} \left\| K(\cdot)\chi_{[0,s]}(\cdot) \right\|_{L^2([-1,0];\mathcal{L}_{HS}(H))}^2 \mathrm{d}s, \end{aligned}$$

where the last step is again due to Lemma 64. Now, for all $s \in [-1,0]$, it holds that

$$\left\| K(\cdot)\chi_{[0,s]}(\cdot) \right\|_{L^2([-1,0];\mathcal{L}_{HS}(H))}^2 \leq \| K(\cdot) \|_{L^2([-1,0];\mathcal{L}_{HS}(H))}^2 ,$$

which gives us

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left(\int_{-1}^{0} \left\| \int_{-1}^{s} e^{2k\pi it} K(t)\mathrm{d}t \right\|_{\mathcal{L}_{HS}(H)} \mathrm{d}s \right)^2 &\leq \int_{-1}^{0} \| K(\cdot) \|_{L^2([-1,0];\mathcal{L}_{HS}(H))}^2 \mathrm{d}s \\ &= \| K(\cdot) \|_{L^2([-1,0];\mathcal{L}_{HS}(H))}^2 < \infty. \end{aligned}$$

The last inequality follows from the assumption on the operator-valued function $K(\cdot) \in L^2([-1,0];\mathcal{L}_{HS}(H))$. This completes the proof of Lemma 70.

□

The following lemma will show that the spectrum of the perturbed operator $\mathcal{A}$ is in some sense similar to the spectrum of the unperturbed operator $\bar{\mathcal{A}}$.

Lemma 71. *Let $L_k = L_0 + 2k\pi i$ be a family of regular bounded curves surrounding the sets $\{\log(\sigma(A)) + 2k\pi i\}_{k\in\mathbb{Z}}$ such that $L_0 \cap \log(\sigma(A)) = \emptyset$ (see Corollary 62) and that the bounded subsets O_k of $\mathbb{C}$ enclosed* *by each L_k are such that* $\mathrm{Int}O_k \cap \mathrm{Int}O_l = \emptyset$ *for* $k \neq l$. *Let the operator-valued functions* $A_{2,3}(\cdot)$ *in* (3.3) *that define the operator* $\mathcal{A}$ *belong to the space* $L^2([-1,0];\mathcal{L}_{HS}(H))$. *Then there exists* $N \in \mathbb{N}_0$ *and a bounded set* $B \subset \mathbb{C}$, *such that* $\sigma(\mathcal{A}) \subset B \bigsqcup_{|k|\geq N} \mathrm{Int}O_k$.

Proof. The existence of such a family of curves L_k follows from the assumption (A2) on the operator A. Note that, due to the Hille-Yosida Theorem (Theorem 15) and the fact that the operator $\mathcal{A}$ generates a C_0-semigroup (Theorem 61), its spectrum is contained in the half-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \lambda_1\}$ for some $\lambda_1 \in \mathbb{R}$. This observation, combined with Corollary 62 shows that the spectrum of the operator $\mathcal{A}$ lies in the vertical strip $\Lambda = \{\lambda \in \mathbb{C} : \lambda_0 \leq \mathrm{Re}(\lambda) \leq \lambda_1\}$ for some $\lambda_0, \lambda_1 \in \mathbb{R}$. Now let M_k be the rectangle $\Lambda \cap \{\lambda \in \mathbb{C} : 2k\pi \leq \ \mathrm{Im}\lambda \leq 2(k+1)\pi\}$ and let $N_k = M_k \setminus \mathrm{Int}O_k$. Due to Corollary 62, the compact sets N_k are subsets of the set $\rho(\bar{\mathcal{A}})$ for $|k| \geq 1$. For $\lambda \in N_k$ let us write $\lambda = \hat{\lambda} + 2k\pi i$, where $\hat{\lambda} \in N_0$. It follows from Proposition 59, that the operator $\Delta^{-1}_{\bar{\mathcal{A}}}(\hat{\lambda} + 2k\pi i) \in \mathcal{L}(H)$ exists for $\hat{\lambda} + 2k\pi i \in N_k$ for all $\hat{\lambda} \in N_0$ and $k \in \mathbb{Z} \setminus \{0\}$. Let us fix $\hat{\lambda} \in N_0$ and write

$$\Delta^{-1}_{\bar{\mathcal{A}}}(\hat{\lambda} + 2k\pi i) = \frac{e^{\hat{\lambda}+2k\pi i}}{\hat{\lambda} + 2k\pi i}\left(A - e^{\hat{\lambda}+2k\pi i}\right)^{-1}, \quad k \in \mathbb{Z} \setminus \{0\}.$$

Since the set N_0 is compact, the continuous functions $\|(A - e^{\hat{\lambda}+2k\pi i})^{-1}\|$ and $|e^{\hat{\lambda}+2k\pi i}|$ are uniformly bounded over k and $\hat{\lambda}$. It follows that

$$\frac{C_1}{|\hat{\lambda} + 2k\pi i|} \leq \left\|\Delta^{-1}_{\bar{\mathcal{A}}}(\hat{\lambda} + 2k\pi i)\right\|_{\mathcal{L}(H)} \leq \frac{C_2}{|\hat{\lambda} + 2k\pi i|}, \tag{3.43}$$

where C_1 and C_2 do not depend on k. Now, due to Lemma 70, we get

$$\begin{aligned}
&\left\|\Delta_{\bar{\mathcal{A}}}(\hat{\lambda} + 2k\pi i) - \Delta_{\mathcal{A}}(\hat{\lambda} + 2k\pi i)\right\|_{\mathcal{L}(H)} \\
&= \left\|\left(\hat{\lambda} + 2k\pi i\right)\int_{-1}^{0} e^{\hat{\lambda}+2k\pi is}A_2(s)\mathrm{d}s + \int_{-1}^{0} e^{\hat{\lambda}+2k\pi is}A_3(s)\mathrm{d}s\right\|_{\mathcal{L}(H)} \leq c_k|\hat{\lambda} + 2k\pi i|,
\end{aligned}$$

where $c_k \to 0$ as $|k| \to \infty$ and c_k does not depend on the choice of $\hat{\lambda} \in N_0$. Hence, due to (3.43),

$$\left\|\Delta_{\bar{\mathcal{A}}}(\hat{\lambda} + 2k\pi i) - \Delta_{\mathcal{A}}(\hat{\lambda} + 2k\pi i)\right\|_{\mathcal{L}(H)} \leq \left\|\Delta^{-1}_{\bar{\mathcal{A}}}(\hat{\lambda} + 2k\pi i)\right\|^{-1}_{\mathcal{L}(H)},$$

for all $\hat{\lambda} \in N_0$ and $|k|$ large enough. Due to Theorem 45 for every $\hat{\lambda} \in N_0$ and $|k| \geq N$, N independent of the choice of $\hat{\lambda}$, the operator $\Delta^{-1}_{\mathcal{A}}(\hat{\lambda} + 2k\pi i)$ exists on N_k. This means (cf. Proposition 59) that, for $|k| \geq N$, the sets N_k are subsets of the set $\rho(\mathcal{A})$. This implies that $\sigma(\mathcal{A}) \subset \bigcup_{|k|\leq N-1} M_k \bigsqcup_{|k|\geq N} \mathrm{Int}O_k$, which completes the proof of Lemma 71.

□

3.3 Main result

Here we prove (Theorem 74) the existence of a Riesz basis of the space M_2 constructed from $\mathcal{A}$-invariant subspaces for the system (3.3) satisfying assumptions (A1) and (A2) and such that that the operator-valued functions $A_{2,3}(\cdot)$ in (3.3) belong to the space $L^2([-1,0];\mathcal{L}_{HS}(H))$. We are thus extending the result from [25] concerning the existence of a Riesz basis of subspaces constructed from Riesz projections to the infinite-dimensional case. An example of an infinite-dimensional space and system for which this results can be applied is given after the proof of Theorem 74. Next lemma will allow us to show that

$$\sum_{|k|\geq N} \left\|\bar{P}_k - P_k\right\|^2_{\mathcal{L}(M_2)} < \infty,$$

for some $N \in \mathbb{N}_0$, where $\bar{P}_k$ and P_k denote the Riesz projections corresponding to the curve L_k for the operators $\bar{\mathcal{A}}$ and $\mathcal{A}$, respectively. The lemma estimates the norm of the difference of resolvents of the operators $\bar{\mathcal{A}}$ and $\mathcal{A}$ on each L_k for $|k|$ large enough by a non-negative sequence belonging to $l^2(\mathbb{Z})$. The idea of using this estimation first appeared in [25].

Lemma 72. *Let L_k be a family of curves as in Lemmas 69 and 71. Let the operator-valued functions $A_{2,3}(\cdot)$ in (3.3) that define the operator $\mathcal{A}$ belong to the space $L^2([-1,0];\mathcal{L}_{HS}(H))$. Then, for some $N \in \mathbb{N}_0$ and $|k| \geq N$, the following estimate holds:*

$$\sup_{\lambda\in L_k} \left\|R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)\right\|_{\mathcal{L}(M_2)} \leq \gamma_k, \quad \textit{where} \quad \sum_{|k|\geq N} {\gamma_k}^2 < \infty.$$

Proof. Due to Lemma 71, the curves L_k are subsets of the set $\rho(\mathcal{A}) \cap \rho(\bar{\mathcal{A}})$ for sufficiently large $|k|$. From the form of the resolvent (3.7) of the operators $\mathcal{A}$ and $\bar{\mathcal{A}}$ it follows that

$$[R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)] \begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix} = \begin{pmatrix} \left(I - Ae^{-\lambda}\right)\left\{\Delta^{-1}_{\mathcal{A}}(\lambda)D_{\mathcal{A}} - \Delta^{-1}_{\bar{\mathcal{A}}}(\lambda)D_{\bar{\mathcal{A}}}\right\} \\ e^{\lambda\theta}\left\{\Delta^{-1}_{\mathcal{A}}(\lambda)D_{\mathcal{A}} - \Delta^{-1}_{\bar{\mathcal{A}}}(\lambda)D_{\bar{\mathcal{A}}}\right\}\end{pmatrix},$$

where

$$\begin{aligned} H \ni D_{\mathcal{A}} &= x + \lambda e^{-\lambda} A \int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s - \int_{-1}^{0} A_2(s)\psi(s)\mathrm{d}s \\ &\quad - \int_{-1}^{0} \left\{\lambda A_2(\theta) + A_3(\theta)\right\} e^{\lambda\theta} \left\{\int_{0}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s\right\} \mathrm{d}\theta, \\ H \ni D_{\bar{\mathcal{A}}} &= x + \lambda e^{-\lambda} A \int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s, \end{aligned}$$

and

$$\mathcal{L}(H) \ni \Delta_{\mathcal{A}}(\lambda) = -\lambda I + \lambda e^{-\lambda} A + \lambda \int_{-1}^{0} e^{\lambda s} A_2(s)\mathrm{d}s + \int_{-1}^{0} e^{\lambda s} A_3(s)\mathrm{d}s, \tag{3.44}$$

$$\mathcal{L}(H) \ni \Delta_{\bar{\mathcal{A}}}(\lambda) = -\lambda I + \lambda e^{-\lambda} A. \tag{3.45}$$

Since $\|(I - Ae^{-\lambda})\|$ and $e^{\lambda\theta}$ are bounded uniformly over k on $L_k \times [-1,0]$, the estimation

$$\sup_{\lambda\in L_k} \left\|\{\Delta^{-1}_{\mathcal{A}}(\lambda)D_{\mathcal{A}} - \Delta^{-1}_{\bar{\mathcal{A}}}(\lambda)D_{\bar{\mathcal{A}}}\}\right\|_H \leq \gamma_k \left\|\begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad \text{where} \quad \sum_{|k|\geq N} {\gamma_k}^2 < \infty \tag{3.46}$$

for some $N \in \mathbb{N}_0$ and $|k| \geq N$ is sufficient to prove Lemma 72. Let us rewrite the difference

$$\left\{\Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}} - \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)D_{\bar{\mathcal{A}}}\right\}$$

as

$$\left\{\Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}} - \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)D_{\bar{\mathcal{A}}}\right\} = \left[\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda)\right] D_{\mathcal{A}} - \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\left[D_{\mathcal{A}} - D_{\bar{\mathcal{A}}}\right]. \tag{3.47}$$

We will first show, that

$$\sup_{\lambda \in L_k} \left\|[\Delta_{\mathcal{A}}^{-1}(\lambda) - \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)]D_{\mathcal{A}}\right\|_H \leq e_k \left\|\begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad \sum_{|k|\geq N} {e_k}^2 < \infty. \tag{3.48}$$

Recall that the operator $\Delta_{\bar{\mathcal{A}}}(\lambda)$ is invertible if and only if $\lambda \notin \bigcup_{k\in\mathbb{Z}} \{\log(\sigma(A)) + 2k\pi i\} \cup \{0\}$, hence for $\lambda \in L_k$ for all $k \in \mathbb{Z}$ we can write

$$\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) = \frac{e^{\lambda}}{\lambda}\left(A - e^{\lambda}\right)^{-1}.$$

Due to the fact that $\|(A - e^{\lambda})^{-1}\|$ and e^{λ} are bounded uniformly over k on each L_k, we get

$$\frac{C_1}{|\lambda|} \leq \left\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)} \leq \frac{C_2}{|\lambda|}, \tag{3.49}$$

where C_1 and C_2 do not depend on k. Now, due to Lemma 70, we get

$$\|\Delta_{\bar{\mathcal{A}}}(\lambda) - \Delta_{\mathcal{A}}(\lambda)\|_{\mathcal{L}(H)} = \left\|\lambda\int_{-1}^{0} e^{\lambda s}A_2(s)\mathrm{d}s + \int_{-1}^{0} e^{\lambda s}A_3(s)\mathrm{d}s\right\|_{\mathcal{L}(H)} \leq c_k|\lambda|$$

for $\lambda \in L_k$, where $c_k \to 0$ as $|k| \to \infty$. Hence, due to (3.49),

$$\|\Delta_{\bar{\mathcal{A}}}(\lambda) - \Delta_{\mathcal{A}}(\lambda)\|_{\mathcal{L}(H)} \leq \left\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)}^{-1} \tag{3.50}$$

holds for $|k|$ large enough. Due to Theorem 45, (3.49), and the fact that $c_k \to 0$ as $|k| \to \infty$, for large enough $|k|$ the operator $\Delta_{\mathcal{A}}^{-1}(\lambda)$ exists for $\lambda \in L_k$ and it holds that

$$\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda)\|_{\mathcal{L}(H)} \leq \frac{\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\|_{\mathcal{L}(H)}}{1 - \|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\|_{\mathcal{L}(H)}c_k} \leq \frac{C_3}{|\lambda|}.$$

From the above considerations and (3.49) we get an estimate of $\left\|\Delta_{\mathcal{A}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)}$

$$\left\|\Delta_{\mathcal{A}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)} \leq \left\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)} + \left\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)} \leq \frac{C_2 + C_3}{|\lambda|} = \frac{C_4}{|\lambda|} \tag{3.51}$$

for some $N_0 \in \mathbb{N}$ and all $|k| \geq N_0$.

Now notice that for the bounded operators $K \in \mathcal{L}(H)$ and $L \in \mathcal{L}(H)$ such that $K^{-1} \in \mathcal{L}(H)$ and $(K + L)^{-1} \in \mathcal{L}(H)$, it holds that

$$K^{-1} - (K + L)^{-1} = (K + L)^{-1}LK^{-1},$$

which is easy to check by multiplying the above equation by K from the right and by $K+L$ from the left. It follows from this consideration, for $\lambda \in L_k$ and $|k|$ large enough, that

$$
\begin{aligned}
&\left(A - e^{\lambda}\right)^{-1} - \left(A - e^{\lambda} + e^{\lambda}\int_{-1}^{0} e^{\lambda s} A_2(s)\mathrm{d}s + e^{\lambda}\lambda^{-1}\int_{-1}^{0} e^{\lambda s} A_3(s)\mathrm{d}s\right)^{-1}\\
&= \left(A - e^{\lambda} + e^{\lambda}\int_{-1}^{0} e^{\lambda s} A_2(s)\mathrm{d}s + e^{\lambda}\lambda^{-1}\int_{-1}^{0} e^{\lambda s} A_3(s)\mathrm{d}s\right)^{-1}\\
&\times \left(e^{\lambda}\int_{-1}^{0} e^{\lambda s} A_2(s)\mathrm{d}s + e^{\lambda}\lambda^{-1}\int_{-1}^{0} e^{\lambda s} A_3(s)\mathrm{d}s\right)\left(A - e^{\lambda}\right)^{-1}.
\end{aligned}
$$

Multiplying the above equality by $\dfrac{e^{\lambda}}{\lambda}$ (see (3.44), (3.45)), gives

$$
\left\|\Delta_{\tilde{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)} = \left\|\Delta_{\mathcal{A}}^{-1}(\lambda)\left(e^{\lambda}\int_{-1}^{0} e^{\lambda s} A_2(s)\mathrm{d}s + e^{\lambda}\lambda^{-1}\int_{-1}^{0} e^{\lambda s} A_3(s)\mathrm{d}s\right)\left[A - e^{\lambda}\right]^{-1}\right\|_{\mathcal{L}(H)}.
$$

Due to (3.51), Lemma 70, and the fact that $\|\left(A - e^{\lambda}\right)^{-1}\|_{\mathcal{L}(H)}$ and e^{λ} are bounded uniformly over k on each L_k, we get

$$
\left\|\Delta_{\mathcal{A}}^{-1}(\lambda) - \Delta_{\tilde{\mathcal{A}}}^{-1}(\lambda)\right\|_{\mathcal{L}(H)} \le \frac{d_k}{|\lambda|}, \quad \text{for } \lambda \in L_k,\ k \ge N_0,\ \sum_{|k|\ge N_0} {d_k}^2 < \infty. \tag{3.52}
$$

Now we will obtain the following estimate

$$
\frac{1}{|\lambda|}\left\|D_{\mathcal{A}}\right\|_H \le C\left\|\begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix}\right\|_{M_2}, \quad \text{for } \lambda \in L_k,\ |k| \ge N_1 \in \mathbb{N}, \tag{3.53}
$$

which, combined with (3.52), will imply

$$
\sup_{\lambda \in L_k} \left\|[\Delta_{\mathcal{A}}^{-1}(\lambda) - \Delta_{\tilde{\mathcal{A}}}^{-1}(\lambda)]D_{\mathcal{A}}\right\|_H \le e_k\left\|\begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix}\right\|_{M_2}, \quad \sum_{|k|\ge N} {e_k}^2 < \infty
$$

for some $N \in \mathbb{N}$ and $|k| \ge N$. To obtain (3.53), we proceed by writing

$$
\begin{aligned}
\frac{1}{|\lambda|}D_{\mathcal{A}} &= \frac{x}{|\lambda|} + \frac{\lambda}{|\lambda|}e^{-\lambda}A\int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s - \frac{1}{|\lambda|}\int_{-1}^{0} A_2(s)\psi(s)\mathrm{d}s\\
&\quad - \frac{\lambda}{|\lambda|}\int_{-1}^{0}\left\{A_2(\theta) + \frac{1}{\lambda}A_3(\theta)\right\}e^{\lambda\theta}\left\{\int_{0}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta.
\end{aligned} \tag{3.54}
$$

The first term on the r.h.s. is clearly bounded by $C_0 \left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2}$. As for the second term, we get

$$\begin{aligned} &\left\| \frac{\lambda}{|\lambda|} e^{-\lambda} A \int_{-1}^{0} e^{-\lambda s} \psi(s) \mathrm{d}s \right\|_H \leq C_1' \left\| \int_{-1}^{0} e^{-\lambda s} \psi(s) \mathrm{d}s \right\|_H \\ &\leq C_1' \int_{-1}^{0} \left\| e^{-\lambda s} \psi(s) \right\|_H \mathrm{d}s \leq C_1 \left\| \psi(\cdot) \right\|_{L^2([-1,0];H)} \leq C_1 \left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2}, \end{aligned} \tag{3.55}$$

where we have used the uniform boundedness of $e^{-\lambda s}$ on $L_k \times [-1, 0]$ over all k and Hölder's inequality. For the third term in (3.54) we get

$$\begin{aligned} &\left\| \frac{1}{|\lambda|} \int_{-1}^{0} A_2(s) \psi(s) \mathrm{d}s \right\|_H \leq \frac{1}{|\lambda|} \int_{-1}^{0} \left\| A_2(s) \right\|_{\mathcal{L}(H)} \left\| \psi(s) \right\|_H \mathrm{d}s \\ &\leq C_2 \left\| \psi(\cdot) \right\|_{L^2([-1,0];H)} \leq C_2 \left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2}, \end{aligned}$$

where the second to last step is again due to Hölder's inequality and the assumption that $A_2(\cdot) \in L^2([-1, 0]; \mathcal{L}_{HS}(H)) \subset L^2([-1, 0]; \mathcal{L}(H))$. As for the last term in (3.54), the following estimate is sufficient

$$\left\| \frac{\lambda}{|\lambda|} \int_{-1}^{0} e^{\lambda\theta} A_2(\theta) \left\{ \int_{0}^{\theta} e^{-\lambda s} \psi(s) \mathrm{d}s \right\} \mathrm{d}\theta \right\|_H \leq C_3 \left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2}. \tag{3.56}$$

We proceed by writing

$$\begin{aligned} &\left\| \frac{\lambda}{|\lambda|} \int_{-1}^{0} e^{\lambda\theta} A_2(\theta) \left\{ \int_{0}^{\theta} e^{-\lambda s} \psi(s) \mathrm{d}s \right\} \mathrm{d}\theta \right\|_H \leq \int_{-1}^{0} \left\| e^{\lambda\theta} A_2(\theta) \left\{ \int_{0}^{\theta} e^{-\lambda s} \psi(s) \mathrm{d}s \right\} \right\|_H \mathrm{d}\theta \\ &\leq \int_{-1}^{0} \left\| e^{\lambda\theta} A_2(\theta) \right\|_{\mathcal{L}(H)} \left\| \int_{0}^{\theta} e^{-\lambda s} \psi(s) \mathrm{d}s \right\|_H \mathrm{d}\theta \leq \int_{-1}^{0} \left\| e^{\lambda\theta} A_2(\theta) \right\|_{\mathcal{L}(H)} \int_{-1}^{0} \left\| e^{-\lambda s} \psi(s) \mathrm{d}s \right\|_H \mathrm{d}\theta \\ &\leq C_3 \left\| \psi(\cdot) \right\|_{L^2([-1,0];H)} \leq C_3 \left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2}, \end{aligned}$$

where we have used the uniform boundedness of $e^{-\lambda s}$ and $e^{\lambda s}$ on $L_k \times [-1, 0]$ for all k, the assumption that $A_2(\cdot) \in L^2([-1, 0]; \mathcal{L}_{HS}(H)) \subset L^2([-1, 0]; \mathcal{L}(H))$ and applied the Hölder's inequality. Combining the above estimates gives (3.53), i.e.,

$$\frac{1}{|\lambda|} \left\| D_{\mathcal{A}} \right\|_H \leq C \left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2}, \quad \lambda \in L_k,\ |k| \geq N_1 \in \mathbb{N}.$$

This, along with (3.52), gives

$$\sup_{\lambda \in L_k} \left\| \left[\Delta_{\tilde{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda) \right] D_{\mathcal{A}} \right\|_H \leq e_k \left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|_{M_2}, \quad |k| \geq N, \quad \sum_{|k| \geq N} e_k^2 < \infty \tag{3.57}$$

for some $N \in \mathbb{N}_0$. From (3.47) we see that in order to prove Lemma 72, it remains to show that

$$\sup_{\lambda\in L_k} \left\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\left[D_{\mathcal{A}}-D_{\bar{\mathcal{A}}}\right]\right\|_H \le f_k \left\|\begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad \sum_{|k|\ge N} {f_k}^2 < \infty. \tag{3.58}$$

Let us rewrite $\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\left[D_{\mathcal{A}}-D_{\bar{\mathcal{A}}}\right]$ in its full form, i.e.,

$$\begin{aligned} &\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\left[D_{\mathcal{A}}-D_{\bar{\mathcal{A}}}\right] \\ &= \frac{e^\lambda}{\lambda}\left(A-e^\lambda\right)^{-1}\left(\int_{-1}^0 A_2(s)\psi(s)\mathrm{d}s - \int_{-1}^0 \left\{\lambda A_2(\theta)+A_3(\theta)\right\}e^{\lambda\theta}\left\{\int_0^\theta e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta\right). \end{aligned} \tag{3.59}$$

For the first term on the r.h.s., due to the uniform over k boundedness of $\left(A-e^\lambda\right)^{-1}$ and e^λ on each L_k, we obtain

$$\begin{aligned} &\left\|\frac{e^\lambda}{\lambda}\left(A-e^\lambda\right)^{-1}\int_{-1}^0 A_2(s)\psi(s)\mathrm{d}s\right\|_H \le \frac{C}{|\lambda|}\int_{-1}^0 \|A_2(s)\|_{\mathcal{L}(H)}\|\psi(s)\|_H\,\mathrm{d}s \\ &\le \frac{C}{|\lambda|}\left(\int_{-1}^0 \|A_2(s)\|^2_{\mathcal{L}(H)}\,\mathrm{d}s\right)^{\frac{1}{2}}\|\psi(\cdot)\|_{L^2([-1,0],H)}. \end{aligned} \tag{3.60}$$

Where the last step was due to Hölder's inequality. Taking into account that for all $\lambda \in L_k$, one has

$$\frac{1}{|\lambda|} \le \frac{1}{2\pi|k|-\hat{C}},$$

with $\hat{C}$ independent of k and combining (3.60) with the assumption $A_2(\cdot) \in L^2([-1,0];\mathcal{L}_{HS}(H)) \subset L^2([-1,0];\mathcal{L}(H))$, we get

$$\left\|\frac{e^\lambda}{\lambda}\left(A-e^\lambda\right)^{-1}\int_{-1}^0 A_2(s)\psi(s)\mathrm{d}s\right\|_H \le f_{1,k}\left\|\begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad \lambda\in L_k, \sum_{k\in\mathbb{Z}} f_{1,k}^2 < \infty. \tag{3.61}$$

As of the estimation of the remaining terms on the r.h.s. of (3.59), it is sufficient to show that

$$\begin{aligned} &\left\|\frac{e^\lambda}{\lambda}\left(A-e^\lambda\right)^{-1}\left[\int_{-1}^0 \lambda A_2(\theta)e^{\lambda\theta}\left\{\int_0^\theta e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta\right]\right\|_H \\ &\le f_{2,k}\left\|\begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad \lambda\in L_k \sum_{k\in\mathbb{Z}} f_{2,k}^2 < \infty. \end{aligned} \tag{3.62}$$

First we note, due to the uniform (over k) boundedness of $\left(A-e^\lambda\right)^{-1}$ and e^λ on each L_k, that the following holds

$$\begin{aligned} &\left\|\frac{e^\lambda}{\lambda}\left(A-e^\lambda\right)^{-1}\left[\int_{-1}^0 \lambda A_2(\theta)e^{\lambda\theta}\left\{\int_0^\theta e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta\right]\right\|_H \\ &\le C\left\|\int_{-1}^0 A_2(\theta)e^{\lambda\theta}\left\{\int_0^\theta e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta\right\|_H. \end{aligned} \tag{3.63}$$

From (3.63), we proceed with

$$
\begin{aligned}
&\int_{-1}^{0} A_2(\theta)e^{\lambda\theta}\left\{\int_{0}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta = -\int_{-1}^{0} A_2(\theta)e^{\lambda\theta}\left\{\int_{\theta}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta \\
&= -\int_{-1}^{0} A_2(\theta)e^{\lambda\theta}\left\{\int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s - \int_{-1}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta \\
&= -\int_{-1}^{0} A_2(\theta)e^{\lambda\theta}\mathrm{d}\theta \int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s + \int_{-1}^{0} A_2(\theta)e^{\lambda\theta}\left\{\int_{-1}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta.
\end{aligned} \tag{3.64}
$$

We estimate the first term in the last line of (3.64) using the uniform (over k) boundedness of $e^{-\lambda s}$ on $L_k \times [-1, 0]$, Hölder's inequality, and Lemma 70, to obtain

$$
\begin{aligned}
&\left\|\int_{-1}^{0} A_2(s)e^{\lambda s}\mathrm{d}s \int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s\right\| \leq f'_{2,k}\left\|\psi(\cdot)\right\|_{L^2([-1,0],H)} \\
&\leq f'_{2,k}\left\|\begin{pmatrix} x \\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \qquad \lambda \in L_k, \sum_{k\in\mathbb{Z}} f'^2_{2,k} < \infty.
\end{aligned} \tag{3.65}
$$

As of the remaining term in the last line of (3.64), by using the integration by parts (see Theorem 36), we obtain

$$
\begin{aligned}
&\int_{-1}^{0} A_2(\theta)e^{\lambda\theta}\left\{\int_{-1}^{\theta} e^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta \\
&= \int_{-1}^{0} A_2(s)e^{\lambda s}\mathrm{d}s \int_{-1}^{0} e^{-\lambda s}\psi(s)\mathrm{d}s - \int_{-1}^{0}\left\{\int_{-1}^{\theta} A_2(s)e^{\lambda s}\mathrm{d}s\right\} e^{-\lambda\theta}\psi(\theta)\mathrm{d}\theta.
\end{aligned} \tag{3.66}
$$

Due to (3.65), we only need to estimate the norm of

$$
\int_{-1}^{0}\left\{\int_{-1}^{\theta} A_2(s)e^{\lambda s}\mathrm{d}s\right\} e^{-\lambda\theta}\psi(\theta)\mathrm{d}\theta.
$$

We proceed with

$$
\begin{aligned}
&\left\|\int_{-1}^{0}\left\{\int_{-1}^{\theta} A_2(s)e^{\lambda s}\mathrm{d}s\right\} e^{-\lambda\theta}\psi(\theta)\mathrm{d}\theta\right\|_H \leq \int_{-1}^{0}\left\|\left\{\int_{-1}^{\theta} A_2(s)e^{\lambda s}\mathrm{d}s\right\} e^{-\lambda\theta}\psi(\theta)\right\|_H \mathrm{d}\theta \\
&\leq \int_{-1}^{0}\left\|\left\{\int_{-1}^{\theta} A_2(s)e^{\lambda s}\mathrm{d}s\right\}\right\|_{\mathcal{L}(H)} \left\|e^{-\lambda\theta}\psi(\theta)\right\|_H \mathrm{d}\theta.
\end{aligned} \tag{3.67}
$$

Due to the uniform boundedness of $e^{-\lambda\theta}$ on $L_k \times [-1, 0]$ for all k and application of the Hölder's inequality, from (3.67) we get

$$
\left\|\int_{-1}^{0}\left\{\int_{-1}^{\theta} A_2(s)e^{\lambda s}\mathrm{d}s\right\} e^{-\lambda\theta}\psi(\theta)\mathrm{d}\theta\right\|_H \leq C\left(\int_{-1}^{0}\left\|\left\{\int_{-1}^{\theta} A_2(s)e^{\lambda s}\mathrm{d}s\right\}\right\|^2_{\mathcal{L}(H)} \mathrm{d}\theta\right)^{\frac{1}{2}} \left\|\psi(\cdot)\right\|_{L^2([-1,0],H)}.
$$

By using similar arguments as in the proof of Lemma 70, we get

$$\begin{aligned}&\left\|\int_{-1}^{0}\left\{\int_{-1}^{\theta}A_2(s)e^{\lambda s}\mathrm{d}s\right\}e^{-\lambda\theta}\psi(\theta)\mathrm{d}\theta\right\|_H \le f''_{2,k}\left\|\psi(\cdot)\right\|_{L^2([-1,0],H)}\\&\le f''_{2,k}\left\|\begin{pmatrix}x\\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad \lambda\in L_k,\ \sum_{k\in\mathbb{Z}}f''^2_{2,k}<\infty.\end{aligned} \tag{3.68}$$

By combining (3.68) with (3.66), (3.65),(3.63) and (3.61) we get the inequality

$$\sup_{\lambda\in L_k}\left\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\left[D_{\mathcal{A}}-D_{\bar{\mathcal{A}}}\right]\right\|_H \le f_k\left\|\begin{pmatrix}x\\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad \sum_{k\in\mathbb{Z}}f_k^2<\infty.$$

The above implies (3.58), i.e.,

$$\sup_{\lambda\in L_k}\left\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\left[D_{\mathcal{A}}-D_{\bar{\mathcal{A}}}\right]\right\|_H \le f_k\left\|\begin{pmatrix}x\\ \psi(\cdot)\end{pmatrix}\right\|_{M_2}, \quad |k|\ge N,\ \sum_{|k|\ge N}f_k^2<\infty,$$

which, combined with (3.57), proves (3.46) and thus Lemma 72.

□

Remark 73. *Note that, due to technical reasons, the number N obtained in the above considerations is somewhat artificial. It follows from the form of the assertion of Lemma 72 that this number can be chosen as the smallest number $\tilde{N}\in\mathbb{N}_0$ for which $L_k\subset\rho(\mathcal{A})$ for $|k|\ge\tilde{N}$.*

Theorem 74, which is the main result of this chapter, uses Lemma 72 and Theorem 24 to show that for system (3.3) satisfying the assumptions (A1) and (A2) and such that the operator-valued functions $A_{2,3}(\cdot)$ in (3.3) belong to the space $L^2([-1,0];\mathcal{L}_{HS}(H))$, there exists a Riesz basis of the space M_2 constructed from $\mathcal{A}$-invariant subspaces, which are the images of Riesz projections of the operator $\mathcal{A}$.

Theorem 74. *Let $L_k = L_0 + 2k\pi i$ be a family of regular bounded curves surrounding the sets $\{\log(\sigma(A)) + 2k\pi i\}_{k\in\mathbb{Z}}$ (see Corollary 62) such that $L_0\cap\log(\sigma(A))=\emptyset$, and that the bounded sets O_k enclosed by each L_k have non-overlapping interiors ($\mathrm{Int}O_k\cap\mathrm{Int}O_l=\emptyset$ for $k\ne l$). Assume that the operator-valued functions $A_{2,3}(\cdot)$ in (3.3) that define the operator $\mathcal{A}$ belong to the space $L^2([-1,0];\mathcal{L}_{HS}(H))$. Then there exists $N\in\mathbb{N}_0$, such that for $|k|\ge N$, $L_k\subset\rho(\mathcal{A})$ and the subspaces of M_2 which are the images of the Riesz projections P_k of the operator $\mathcal{A}$ associated with the curves L_k, together with the image of the orthogonal projection P_α to the subspace $\overline{\mathrm{span}\{P_kM_2,\,|k|\ge N\}}^{\perp}$, constitute a Riesz basis of subspaces of the space M_2.*

Proof. First note that such a family L_k exists by the assumption (A2) on the operator A. Due to Lemma and 72, for $\bar{P}_k$ as in Lemma 69, it holds for some $N\in\mathbb{N}_0$ for $|k|\ge N$, that $L_k\subset\rho(\mathcal{A})\cap\rho(\bar{\mathcal{A}})$

and

$$\left\|P_k - \bar{P}_k\right\|_{\mathcal{L}(M_2)} = \left\| \oint\limits_{L_k} R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)\mathrm{d}\lambda \right\|_{\mathcal{L}(M_2)} \leq \oint\limits_{L_k} \left\|R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)\right\|_{\mathcal{L}(M_2)} |\mathrm{d}\lambda| \leq |L_k| \sup_{\lambda\in L_k} \left\|R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)\right\|_{\mathcal{L}(M_2)} \leq |L_0|\gamma_k, \tag{3.69}$$

where $\sum\limits_{|k|\geq N} {\gamma_k}^2 < \infty$, and $|L_0|$ denotes the length of the curve L_0. For $\{\bar{Q}_k\}_{k\in\mathbb{Z}}$ as in Lemma 69 and $|k| \geq N$, we get

$$\left\|\bar{Q}_k - P_k\right\|_{\mathcal{L}(M_2)} \leq \left\|\bar{Q}_k - \bar{P}_k\right\|_{\mathcal{L}(M_2)} + \left\|P_k - \bar{P}_k\right\|_{\mathcal{L}(M_2)}. \tag{3.70}$$

Due to (3.38) and (3.69), it follows

$$\sum_{|k|\geq N} \left\|\bar{Q}_k - P_k\right\|^2_{\mathcal{L}(M_2)} < \infty. \tag{3.71}$$

Let us now introduce a new scalar product $\langle\cdot,\cdot\rangle_{M_{2,1}}$ on M_2 which equips the space M_2 with a norm equivalent to the initial one (cf. (3.11)) by

$$\left\langle \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix}, \begin{pmatrix} y \\ \phi(\cdot) \end{pmatrix} \right\rangle_{M_{2,1}} \equiv \left\langle \begin{pmatrix} x \\ A^{-(\cdot)}\psi(\cdot) \end{pmatrix}, \begin{pmatrix} y \\ A^{-(\cdot)}\phi(\cdot) \end{pmatrix} \right\rangle_{M_2}$$

The norm equivalence comes from the fact that the bounded operator $A^{-(\cdot)}$ on $L^2([-1,0];H)$ is invertible with a bounded inverse $A^{(\cdot)}$. This can be seen by writing (cf. (3.12))

$$\left\| \begin{pmatrix} x \\ \psi(\cdot) \end{pmatrix} \right\|^2_{M_{2,1}} = \|x\|^2_H + \|A^{-(\cdot)}\psi(\cdot)\|^2_{L^2([-1,0];H)}.$$

It follows from the form of subspaces $\{\bar{Q}_k M_2\}_{k\in\mathbb{Z}}$, which form a complete set of subspaces (see (3.34)), that with respect to this new scalar product these subspaces are mutually orthogonal and the projections $\{\bar{Q}_k\}_{k\in\mathbb{Z}}$ are orthogonal.

Now, let $\widetilde{Q}_0 = \sum\limits_{|k|<N} \bar{Q}_k$, $\widetilde{P}_0 = P_\alpha$. For $k > 0$ denote $\widetilde{Q}_k = \bar{Q}_{k+(N-1)}$ and $\widetilde{P}_k = P_{k+(N-1)}$, and for $k < 0$ denote $\widetilde{Q}_k = \bar{Q}_{k-(N-1)}$ and $\widetilde{P}_k = P_{k-(N-1)}$. From (3.71) we obtain

$$\sum_{k\in\mathbb{Z}} \left\|\widetilde{Q}_k - \widetilde{P}_k\right\|^2_{\mathcal{L}(M_{2,1})} < \infty. \tag{3.72}$$

Observe that subspaces $\{\widetilde{Q}_k M_2\}_{k\in\mathbb{Z}}$ form an orthogonal base of subspaces for the space M_2 endowed with the new norm. Now, since for any $x \in M_2$ it holds that

$$x = P_\alpha x + (I - P_\alpha)x,$$

and the subspaces $\{P_k M_2\}_{|k|\geq N}$ are complete in $(I - P_\alpha)M_2 = \overline{\operatorname{span}\{P_k M_2,\, |k| \geq N\}}$, the set of subspaces $\left\{\widetilde{P}_k M_2\right\}_{k\in\mathbb{Z}}$ forms a complete set of subspaces in M_2 w.r.t. both the $\|\cdot\|_{M_{2,1}}$ norm and the original norm. Also, for each k, the minimal angle between the subspace $\widetilde{P}_k M_2$ and the closed linear hull of the rest of subspaces $\widetilde{P}_j M_2$, $(j \neq k)$ is positive w.r.t the new norm. This holds due to the application of Lemma 66 followed by Remark 67, Lemma 68 and the observation

$$\begin{aligned} 0 = P_j P_k = P_k P_j, \quad &\text{for all } |k|, |j| \geq N,\ j \neq k, \\ P_\alpha \perp \overline{\operatorname{span}\{P_k M_2,\ |k| \geq N\}}, \end{aligned} \tag{3.73}$$

which is true since the operators P_k, $|k| \geq N$, are Riesz projections that correspond to disjoint parts of the spectrum and due to the definition of P_α,

Since the norm given by the new scalar product $\langle\cdot,\cdot\rangle_{M_{2,1}}$ is equivalent to the original norm, it follows from (3.72) and Lemma 65 that the subspaces generated by the projection operators $\{\widetilde{P}_k\}_{k\in\mathbb{Z}}$ are quadratically close to a complete set of orthogonal subspaces generated by $\{\widetilde{Q}_k\}_{k\in\mathbb{Z}}$ w.r.t. the new norm. The fact that the projection operators $\{\widetilde{P}_k\}_{k\in\mathbb{Z}}$ generate a Riesz basis of subspaces of the space M_2 equipped with the norm $\|\cdot\|_{M_{2,1}}$, and thus w.r.t. the original norm (since the norms are equivalent, see Theorem 20), follows from the observation that the set of subspaces $\{\widetilde{P}_k M_2\}_{k\in\mathbb{Z}}$ is complete, Theorem 24 followed by Remark 26, and the fact that, for each k, the minimal angle between the subspace $\widetilde{P}_k M_2$ and the closed linear hull of the rest of subspaces $\widetilde{P}_j M_2$, $(j \neq k)$ is positive (see (3.73)).

Note that, for $|k| \geq 1$ the subspaces $\widetilde{P}_k M_2$, being the images of Riesz projections of the opearator $\mathcal{A}$, are $\mathcal{A}$-invariant (see Definition 16).

□

We end this chapter by providing an example of a Hilbert space for which our results may be applicable.

Remark 75. *Consider as the space H the space $L^2[0,1]$, which is a separable Hilbert space, and the delay equation of neutral type of the form*

$$\dot{z}(s,t) = A\dot{z}(s,t-1) + \int_{-1}^{0}\int_{0}^{1} k_2(s,u,\theta)\dot{z}(u,t+\theta)\mathrm{d}u\mathrm{d}\theta + \int_{-1}^{0}\int_{0}^{1} k_3(s,u,\theta)z(u,t+\theta)\mathrm{d}u\mathrm{d}\theta, \tag{3.74}$$

where $z(s,t) \in L^2[0,1]$ for all $t \geq 0$ and A is a bounded invertible operator on $L^2[0,1]$ satisfying the assumption (A2) (e.g., $(Af(\cdot))(s) = f(s) + \int_0^s f(u)\mathrm{d}u$)[2] and the integral kernels are such that

$$\int_0^1\int_0^1 |k_{2,3}(s,u,\theta)|^2\,\mathrm{d}u\mathrm{d}s < \infty$$

for all $\theta \in [-1,0]$, and

$$\int_{-1}^0\int_0^1\int_0^1 |k_{2,3}(s,u,\theta)|^2\,\mathrm{d}u\mathrm{d}s\mathrm{d}\theta < \infty,$$

which renders the integral operators defined by the kernels $k_{2,3}(s,u,\theta)$ Hilbert-Schmidt operators such that their Hilbert-Schmidt norm is square-integrable over $\theta \in [-1,0]$ (see Section 1.4.1 in Chapter 1).

[2] Recall that for the Volterra operator $(Vf(\cdot))(s) = \int_0^s f(u)\mathrm{d}u$, $\sigma(V)$ is equal to the singleton set $\{0\}$.

Then for the system (3.74) the corresponding C_0-semigroup generator is of the form:

$$\mathcal{A}_{L^2[0,1]}\begin{pmatrix} y \\ z(\cdot)\end{pmatrix} = \begin{pmatrix} \displaystyle\int_{-1}^{0}\int_{0}^{1} k_2(s,u,\theta)\dot{z}(u,\theta)\mathrm{d}u\mathrm{d}\theta + \int_{-1}^{0}\int_{0}^{1} k_3(s,u,\theta)z(u,\theta)\mathrm{d}u\mathrm{d}\theta \\ \mathrm{d}z(\theta)/\mathrm{d}\theta \end{pmatrix},$$

where the domain of the operator $\mathcal{A}_{L^2[0,1]}$ is given by

$$D(\mathcal{A}_{L^2[0,1]}) = \{(y, z(\cdot)) : z(\cdot) \in H^1\left([-1,0]; L^2[0,1]\right), y = z(0)-Az(-1)\} \subset L^2[0,1]\times L^2\left([-1,0]; L^2[0,1]\right).$$

Due to Theorem 74 for this system there exists a Riesz basis of the space $M_{2,L^2[0,1]}$ constructed from $\mathcal{A}_{L^2[0,1]}$-invariant subspaces of the form as in Theorem 74.

Bibliography

[1] W. Arendt and C.J.K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Am. Math. Soc. **306** (1988), 837–852.

[2] C.J.K. Batty, *Asymptotic behaviour of semigroups of operators*, Banach Cent. Publ. **30** (1994), 35–52.

[3] C.J.K. Batty, A. Borichev, Y. Tomilov, *Lp-tauberian theorems and Lp-rates for energy decay*, J. Funct. Anal. **270** (2016), 1153–1201.

[4] C.J.K. Batty and T. Duyckaerts, *Non-uniform stability for bounded semi-groups on Banach spaces*, J. Evol. Equ. **8** (2008), 765–780.

[5] C.J.K. Batty and S. Yeates, *Weighted and local stability of semigroups of operators*, Math. Proc. Camb. **129** (2000), 85–98.

[6] A. Borichev and Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. **347** (2010), 455–478.

[7] J. Burns, T. Herdman, H. Stech. *Linear functional-differential equations as semigroups on product spaces*, SIAM J. Math. Anal. **14** (1983), 98–116.

[8] T. Bühler, D.A. Salamon, *Functional Analysis*, J. Am. Math. Soc., (2018).

[9] B.D. Craven, *Two properties of Bochner integrals*, Bull. Ausrtal. Math. Soc. **3** (1970), 363–368.

[10] K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer, New York (2000).

[11] I.M. Gel'fand, *Remark on the work of N. K. Bari "Biorthogonal systems and bases in Hilbert space"*, Moskov. Gos. Univ. Ucen. Zap. **148**, Matematika 4 (1951), 224–225 (In Russian).

[12] I.C. Gohberg, M.G. Krein, *Introduction to the theory of linear non-selfadjoint operators in Hilbert space*, Nauka, Moscow (1965) (in Russian).

[13] F. Huang, *Strong asymptotic stability of linear dynamical systems in Banach spaces*, J. Differ. Equ. **104** (1993), 307–324.

[14] T. Hytönen, J. van Neerven, M. Veraar, L. Weis *Analysis in Banach spaces, Vol 1: Martingales and Littlewood-Paley Theory*, Springer Cham (2016).

[15] L.A. Lusternik, V.J. Sobolev, *Elements of Functional Analysis*, Pań
stowe Wydawnictwo Naukowe, Wrocław (1959) (in Polish).

[16] Y. Lyubich and V. Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Stud. Math. **88** (1988), 37–42.

[17] A.S. Markus, *A basis of root vectors of a dissipative operator*, Dokl. Akad. Nau SSSR **13**(2) (1960), 524–527 = Soviet Math. Dokl. **1**, (1960) 599–602.

[18] J. van Neerven, *The asymptotic behaviour of semigroups of linear operators*, Birkhäuser, Basel (1996).

[19] A. Pazy *Semigroups of Linear Operators and Applications to Partial Differential Equations* Springer New York, New York (1983).

[20] V. Phóng, *Semigroups with nonquasianalytic growth,* Stud. Math., **104** (1993), 229–241.

[21] V. Phóng, *Theorems of Katznelson-Tzafriri type for semigroups of operators*, J. Funct. Anal. **103** (1992), 74–84.

[22] R. Rabah, G.M. Sklyar, P.Y. Barkhayev, *Exact null controllability, complete stabilizability and continuous final observability of neutral type systems*, Int. J. Appl. Math. Comput. Sci. **27** (3): (2017) 489–499.

[23] R. Rabah, G.M. Sklyar, P.Y. Barkhayev, *Stability and stabilizability of mixed retarded-neutral type systems*, ESAIM: COCV **18** (3) (2012), 656–692.

[24] R. Rabah, G.M. Sklyar, A.V. Rezounenko, *On Pole Assignment and Stabilizability of Neutral Type Systems*, J. J. Loiseau, W. Michiels, S.I. Niculescu, R. Sipahi, Topics in Time Delay Systems, Springer (2009), 85–93.

[25] R. Rabah, G.M. Sklyar, A.V. Rezounenko, *Stability analysis of neutral type systems in Hilbert space*, J. Differ. Equ., **214** (2005), 391–428.

[26] G.M. Sklyar. *Lack of maximal asymptotics for linear differential equations in Banach spaces*, Dokl. Math. **81** (2010), 265–267.

[27] G.M. Sklyar, *On the decay of bounded semigroup on the domain of its generator*, Vietnam J. Math. **43** (2015), 207–213.

[28] G.M. Sklyar, *On the maximal asymptotics for linear differential equations in Banach spaces*, Taiwan. J. Math. **14** (2010), 2203–2217.

[29] G.M. Sklyar and V. Marchenko, *Hardy inequality and the construction of infinitesimal operators with non-basis family of eigenvectors*, J. Funct. Anal. **272** (2017), 1017–1043.

[30] G.M. Sklyar, V. Marchenko, and P. Polak, *Sharp polynomial bounds for certain C_0-groups generated by operators with non-basis family of eigenvectors*, J. Funct. Anal. **280** (2021).

[31] G.M. Sklyar and P. Polak, *Notes on the asymptotic properties of some class of unbounded strongly continuous semigroups*, J. Math. Phys. Anal. Geom. **15** (2019), 412–424.

[32] G.M. Sklyar and P. Polak, *On asymptotic estimation of a discrete type C_0-semigroups on dense sets: Application to neutral type systems*, Appl. Math. Optim. **75** (2017), 175–192.

[33] G.M. Sklyar, P. Polak, and B. Wasilewski, *On the relative decay of unbounded semigroups on the domain of the generator* – accepted for publication in Journal of Mathematical Physics, Analysis, Geometry.

[34] G.M. Sklyar and V. Shirman, *On asymptotic stability of linear differential equation in Banach spaces*, Teoria Funk., Funkt. Anal. Prilozh. **37** (1992), 127–132 (in Russian).

[35] P.M. Sołtan, *Elements of operator theory in Hilbert spaces*, Wydawnictwo Uniwersytetu Warszawskiego, Warsaw (2018) (in Polish).

[36] J. Woźniak, M. Firkowski, *Optimal decay ratio of damped slowly rotating Timoshenko beams*, ZAMM Z. Angew. Math. Mech. **99** (2019).

[37] G.Q. Xu and S.P. Yung, *The expansion of a semigroup and a Riesz basis criterion*, J. Differ. Equ. **210** (2005), 1–24.

[38] K. Yosida, *Functional analysis*, Springer-Verlag, (1965).

[39] H. Zwart, *Riesz basis for strongly continuous groups*, J. Differ. Equ. **249** (2010), 2397–2408.