REPORT ON THE DISSERTATION "ASYMPTOTICS ANALYSIS AND ANALYTIC TOOLS FOR CERTAIN TYPES OF C_0-SEMIGROUPS" BY MGR. BARTOSZ WASILEWSKI

DR HAB. JAN ROZENDAAL
PROF. IMPAN
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

General information. The dissertation "Asymptotics analysis and analytic tools for certain types of C_0-semigroups," by Mgr. Bartosz Wasilewski, concerns the theory of evolution equations and C_0-semigroups of operators. Its contributions involve two relatively disparate topics: the asymptotic behavior of C_0-semigroups, and Riesz bases associated with delay differential equations.

The dissertation was written at the University of Szczecin, under the supervision of Prof. dr hab. Grigory M. Sklyar (the auxiliary supervisor is Dr. Piotr Polak). It contains 69 pages, the mathematical content of which comprises 57 pages, divided into three chapters.

Other work by Mgr. Wasilewski in mathematics is as follows:

(1) G. M. Sklyar, P. Polak and B. Wasilewski, *On the relative decay of unbounded semigroups on the domain of the generator*. Accepted for publication in Journal of Mathematical Physics, Analysis, Geometry.
(2) G. M. Sklyar, P. Polak and B. Wasilewski, *On the extension of battys theorem on the semigroup asymptotic stability*. Available at https://doi.org/10.48550/arXiv.2112.01233, 2021.
(3) G. M. Sklyar, P. Polak and B. Wasilewski, *Some Notes on the Asymptotic Behavior of Unbounded Semigroups on the Domain of the Generator*. 2023 31st Mediterranean Conference on Control and Automation (MED), Limassol, Cyprus, 2023, pp. 989-993.
(4) G. M. Sklyar and B. Wasilewski, *The Riesz basis property of infinite-dimensional delay differential equations*. In preparation.

The articles (1), (3) and (4) were mentioned in the documentation accompanying the dissertation; (2) was not mentioned there but is available online. Unfortunately I have not been able to access (1), (3) and (4). In the case of (1), the article does not appear to have been published yet, and I was not sent a link to a preprint version of the article. In the case of (3), there does not appear to be a freely accessible preprint version of the article online, and I was not able to access the relevant conference proceedings through my home institution. Finally, at the time of writing, I could not locate an online version of (4).

It is indicated in the dissertation that Chapter 2 is based on results from (1). On the other hand, an inspection of (2) shows that the contents of Chapter 2 overlap to a very large extent with those of (2), leading me to speculate that (2) might be a preprint version of (1). On the other hand, the title and abstract of (3) indicate that its contents are also related to those in (1) and (2).

It would have been highly desirable for (1), (3) and (4) to have been made available to the reviewer (and more generally, in the case of (1) and (3), to the wider scientific community). Moreover, the connection between (2) and the work in the dissertation should have been clarified. Without such clarification, I have been forced to speculate as to the connection between these articles and the dissertation.

It should also be noted that I did not receive any statements by co-authors regarding the contributions of Mgr. Wasilewski to the articles listed above, or regarding the work in the dissertation. Hence I am also forced to speculate regarding the contributions of Mgr. Wasilewski in this regard.

Contents of the dissertation. The dissertation concerns evolution equations of the form

$$\dot{u}(t) = Au(t), \quad t \geq 0, \tag{0.1}$$

for $u : [0, \infty) \to X$ a function which assumes value in a Banach space X, and A an (in general unbounded) operator on X. The abstract Cauchy problem associated with such an equation naturally leads to the theory of strongly continuous operator semigroups (C_0-semigroups) on X. This theory has a long and successful history (see e.g. [2, 5]), and it has become a systematic framework for the study of evolution equations.

From its onset, a central topic of interest in the theory of evolution equations has been the asymptotic behavior of C_0-semigroups, which in turn corresponds to the long-term behavior of solutions to (0.1) (see e.g. [8]). Foundational results concerning the strong stability of C_0-semigroups include, among others, [1, 6]. Over the past 15 years or so, attention has shifted to so-called *semi-uniform decay* (see e.g. [3, 4]). For a C_0-semigroup $(T(t))_{t\geq 0}$ with invertible generator A, this phenomenon revolves around norm estimates for $T(t)A^{-1}$, as $t \to \infty$. In turn, such estimates are intimately connected to spectral properties of the operator A, as is already evident in the finite-dimensional setting. The first main contribution of the dissertation, in Chapter 2, considers semi-uniform decay under suitable spectral assumptions on A.

Typically, the asymptotic theory of C_0-semigroup concerns itself with deriving statements about the asymptotic behavior of a semigroup $(T(t))_{t\geq 0}$ from spectral properties of its generator A. In the case where the underlying Banach space X is a Hilbert space which has an orthonormal basis of eigenvectors of A, this problem simplifies enormously. A somewhat less restrictive, but still very useful, condition is that X has a Riesz basis of subspaces which are invariant under the action of A. The second contribution of the thesis, in Chapter 3, concerns Riesz bases of invariant subspaces associated with a specific type of evolution equation, arising from the study of delay differential equations.

Finally, Chapter 1 is of an introductory nature and collects the background required for the rest of the dissertation.

Chapter 1. This chapter is divided into five sections: one on spectral theory and C_0-semigroups, the second on Riesz projections and Riesz bases of subspaces, another one on Bochner integrals, a fourth one on Hilbert–Schmidt operators, and finally a section containing some auxiliary tools.

The first section contains some basic material on the spectrum of unbounded operators on Banach spaces, and on asymptotics for C_0-semigroups and the associated evolution equations. Some very basic facts and definitions are collected here, which allows the reader to conveniently look them up later on in the dissertation.

The second section contains material on Riesz bases of subspaces of Hilbert spaces. These definitions and results are definitely less likely to be found in the everyday toolbox of researchers in evolution equations. Nonetheless, this section is quite important, because Riesz projections and Riesz bases of subspaces actually appear in both Chapters 2 and 3, thereby providing a link between the two main contributions of the thesis.

The next two sections, on Bochner integration and Hilbert–Schmidt operators, contain a few very basic definitions and results on these topics.

The final section introduces Sobolev spaces of functions that assume values in a Banach space, together with the Riesz–Dunford holomorphic functional calculus for bounded operators on a Banach space X.

In my opinion, this first chapter is well written and conveniently organized. Most of the results and definitions are very basic, but it is useful to have them all easily available when reading the rest of the dissertation.

Chapter 2. As indicated above, this chapter is concerned with semi-uniform decay of a C_0-semigroup $(T(t))_{t\geq 0}$ with generator A on a Banach space X, i.e. with the behavior of $\|T(t)R(\lambda, A)\|$ as $t \to \infty$. Here $R(\lambda, A) := (\lambda - A)^{-1}$ is the resolvent at a given point $\lambda \in \rho(A) := \mathbb{C} \setminus \sigma(A)$.

It is known from [3] that, if $(T(t))_{t\geq 0}$ is uniformly bounded, then one has $\|T(t)R(-1, A)\| \to 0$ if and only if $\sigma(A) \cap i\mathbb{R} = \emptyset$. On the other hand, if $(T(t))_{t\geq 0}$ is not uniformly bounded, then for some $\lambda \in \rho(A)$ one may still have $\|T(t)R(\lambda, A)\| \to 0$ as $t \to \infty$, even if $\sigma(A) \cap i\mathbb{R} \neq \emptyset$.

It is also of interest to obtain more general estimates for $\|T(t)R(\lambda, A)\|$ as $t \to \infty$, e.g. statements of the form

$$\frac{\|T(t)R(\lambda, A)\|}{\alpha(t)} \to 0 \text{ as } t \to \infty, \tag{0.2}$$

for a suitable $\alpha : [0, \infty) \to [0, \infty)$. One such statement is contained in [7]. It says that, if $\sigma(A) \cap i\mathbb{R} = \emptyset$ and if α has a reduced weight which is not quasi-analytic (see Section 2.1 of the dissertation for definitions) and satisfies $\|T(t)\| \leq \alpha(t)$ for all $t \geq 0$, then (0.2) holds. In fact, [7] mostly concerns bounds for the Hille–Phillips functional calculus for A, and (0.2) is a specific instance of such a bound.

The main result in Chapter 2 of the dissertation, Theorem 52, extends parts of [3, 7] by obtaining (0.2) for unbounded semigroups, without the assumption that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Loosely speaking, the main theorem says that, if for each $\lambda \in \sigma(A) \cap i\mathbb{R}$ the spectral projection P_λ associated with λ is well defined through the holomorphic functional calculus and satisfies

$$\lim_{t\to\infty} \frac{\|T(t)P_\lambda\|}{\alpha(t)} = 0,$$

then (0.2) holds. Here α satisfies a suitable growth condition and is such that $\|T(t)\| \leq \alpha(t)$ for all $t \geq 0$. This should of course be compared with the special case where X is a Hilbert space and the P_λ are projections onto eigenspaces associated with an orthonormal basis of eigenvectors for A. In fact, the examples used in Section 2.3 to illustrate the main theorem arise from the latter setting by (possibly) changing the norm of the underlying Hilbert space.

The proof of Theorem 52 relies on an interesting operator-theoretic construction, involving a semigroup $(\widetilde{T}(t))_{t\geq 0}$ associated with $(T(t))_{t\geq 0}$ but acting on a subspace $\widetilde{X}$ of the space $\mathcal{L}(X)$ of bounded operators on X, as well as a nontrivial norm on a quotient space of $\widetilde{X}$. For me, this proof is the highlight of the dissertation. However, it should be noted that the ideas in the proof have appeared in the literature before, including in work by Mgr. Wasilewski's two supervisors. Hence it is not clear to me to what extent the proof was conceived by Mgr. Wasilewski.

Finally, Section 2.3 contains two examples that are meant to illustrate the strength of Theorem 52. One of them is quite straightforward, while the second one requires more calculations, at least if one attempts to prove suitable decay estimates without the use of Theorem 52. It should be noted that it is not completely clear whether a different calculation "by hand" than the one used by the authors would also work and be simpler, but on the other hand it definitely seems reasonable to expect that Theorem 52 helps to prove decay estimates here. Another remark is that, although these examples are simple to state and elegant from an operator-theoretic perspective, they do not seem to arise from concrete partial differential equations. This is a bit of a drawback, given that some of the most powerful applications of semi-uniform decay estimates concern partial differential equations (e.g. damped wave equations). It would have been very interesting if one could find a concrete application of the results in this chapter.

Chapter 3. This chapter deals with a delay differential equation of the following form:

$$\dot{z}(t) = A\dot{z}(t) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta), \quad t \geq 0. \tag{0.3}$$

Here $z : [-1,\infty) \to X$ satisfies $\mathbf{1}_{[-1,0]}z \in W^{1,p}([-1,0];X)$ for X a Banach space and $p \in [1,\infty)$, and one has $A_2, A_3 \in L^q([-1,0];\mathcal{L}(X))$ for q the Hölder conjugate of p. One can reformulate (0.3) in a convenient (and not completely trivial) manner as an evolution equation, which brings it within the scope of the techniques used in the thesis. In particular, for X a Hilbert space one can can consider Riesz bases of subspaces which are inariant under the generator of the associated semigroup.

The main result of this chapter is the existence of such a basis. As mentioned before, the existence of a Riesz basis of invariant subspaces can allow one to deduce asymptotic properties of the associated evolution equation. This, together with the use of Riesz projections in the proofs, seems to be the most concrete link between the two problems considered in the dissertation. On the other hand, no application of the results in Chapter 3 to decay estimates for an equation of the form (0.3) is given. Remark 75 does provide an example to which the main result of this chapter is applicable, but no asymptotic properties are considered.

After stating the problem in Section 3.1, in Section 3.2 some preliminary results are proved, such as the fact that (0.3) is well posed as an evolution equation. Some basic spectral data of the generator $\mathcal{A}$ of the associated C_0-semigroup is also determined, including a formula for the resolvent of $\mathcal{A}$. From then on X is assumed to be a Hilbert space, and Riesz bases of subspaces from Chapter 1 are brought into the picture. Various properties of the projections onto such subspaces are determined. Here it should be noted that these projections are again given by the holomorphic functional calculus, and these are well defined due to the nature of the spectrum of $\mathcal{A}$. Although this section contains preliminary results, it is actually the longest section of the dissertation, and much of the hard work in Chapter 3 takes place here.

On the other hand, Section 3.3, where the main result of Chapter 3 is proved, also still involves quite a bit of work. Throughout, the key point of the proof appears to be a comparison between the properties of the resolvent of $\mathcal{A}$ and the resolvent of the operator $\tilde{\mathcal{A}}$ which is associated with (0.3) in the case where $A_2 = A_3 \equiv 0$. The latter is much easier to work with, and $\mathcal{A}$ inherits various properties from $\tilde{\mathcal{A}}$ through perturbation. On the other hand, this is definitely not the only ingredient of the proof, which is more involved than the proof of the main result in Chapter 2, in my opinion.

In fact, by itself Chapter 3 is quite impressive, and the mathematics used here is both nontrivial and requires hard work. On the other hand, it is crucial to note that the same statement was proved in the finite-dimensional case, where $X = \mathbb{C}^n$, by the PhD candidate's supervisor, and in that case a more thorough analysis was performed. Since it is relatively common (although definitely not automatic) that problems in finite dimensions extend to general separable Hilbert spaces, one is naturally led to wonder which of the ideas in this chapter are due to Mgr. Wasilewski.

Evaluation of the dissertation. In my opinion, the mathematical content of the thesis is of a sufficiently high level for a PhD degree. The results as such are probably not groundbreaking, but the techniques that go into the proofs are nontrivial, and some of them are nonstandard.

On the other hand, it is not clear to me to what extent the ideas that underlie the proofs are due to Mgr. Wasilewski. As indicated above, the main ideas have appeared in the literature before, and even in work of the PhD candidate's supervisors.

Moreover, given that other mathematical work of Mgr. Wasilewski, such as the articles on which the dissertation is based, have not been made available to me and are not widely available to the mathematical community, it is difficult to determine to what extent the thesis differs from the articles on which it is based. Here it should be noted that preprint (2), which was not mentioned in the documentation accompanying the dissertation, is almost identical in content and format to Chapter 2 of the thesis. Hence it seems that the latter chapter was taken directly, without many modifications, from this preprint. By itself, this is not problematic and in fact common for PhD theses, but the lack of clarity regarding the other work of the PhD candidate is definitely somewhat confusing.

I do find the dissertation well written, and it contains very few typos.

Conclusion. The submitted thesis fulfills the conditions set by Article 187 ust. 1-3 ustawy z dnia 20 lipca 2018 r. Prawo o szkolnictwie wyższym i nauce (Dz. U. z 2023 r. poz. 742 ze zm.). Hence I recommend that Mgr. Wasilewski be allowed to proceed to the next stage of the process of obtaining a PhD degree in the field of mathematics.

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Jan Rozendaal